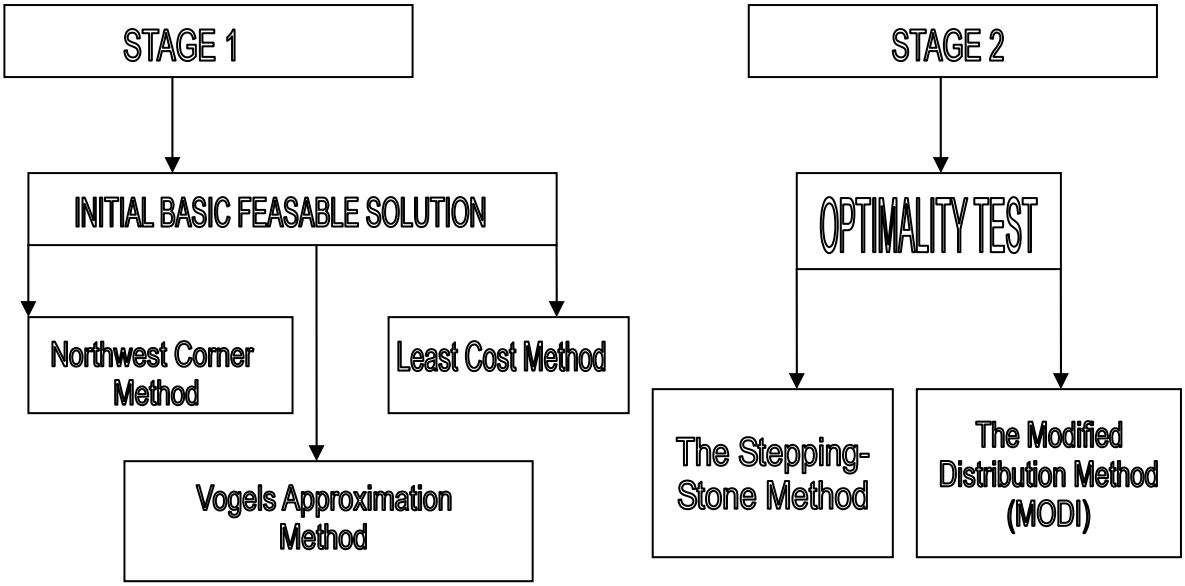
11

The Transportation Problem

Stages in solving transportation problems



Steps for Stage 1 (IBFS)

1) If problem is not balanced, introduce dummy row/column to balance it.
   [***Note:*** Balancing means ensuring that the total of availability and requirement is equal. If total is not equal then we need to make them equal, by introducing dummy row/column.]
2) In case of problems of maximization, Convert profit matrix into loss matrix. This is done by subtracting each entry in the table from largest no. in the table.
3) Assign M or ∞ to prohibited routes (i.e. routes which are not available) etc.
4) Find Initial Basic Feasible Solution using methods defined below.

[Important Note: 1. Though we can also use step 2 before step 1 (i.e. we can also convert profit matrix in loss matrix before balancing the problem), but, it is recommended to apply the steps in above defined order only.
2. If question is silent, we need to find IBFS using VAM (because it provides least of all values of transportation cost found by different method.]

Steps in Northwest Corner Method

1. Allocation always starts from upper left hand corner (i.e. North side first row-first column). Assignment is made in such a way that the resources available are exhausted or demand is fully satisfied.
2. If the resources available are fully exhausted then we move down to second next row and continue

the process till the whole demand is exhausted.
3. If the first allocation completely satisfies the demand then we move to the next column of the same row, and continue the allocation process till all availability and requirements are met.
4. The procedure is repeated till all row availability and column requirements are met.

Steps in Least Cost Method

1. Allocation always starts from the cell whose transportation cost per unit is least.
2. The lowest cell is filled as much as possible in view of the availability and destination requirement of its column.
3. Then we move to the next lowest cell and so on continue the procedure in view of the remaining availability of demand and supply.
4. The procedure is repeated till all row availability and column requirements are met.
5. In case of tie for the lowest cell during allocation, choice may be made for a row or column by which maximum requirement is exhausted.

Steps in Vogel's method

1. Find the difference between two least cost cells in every row and column.
2. Identify the row or column with the highest of the difference. It is in this row or column where allocation should be made.
3. In the row or column selected in step 2, identify the least cost cell. It is in this cell allocation should be made.
4. If there is a tie amongst the largest differences, the choice may be made for a row or column which has least cost. In case there is a tie in cost cell also, choice may be made for a row or column by which maximum requirement is exhausted.
5. Hatch that row or column containing this cell whose totals have been exhausted so that this row or column is ignored in further consideration.
6. Re-compute the row & column differences for the reduced TLP go ahead to Step 3. Continue the above steps till all allocations are made.

Question 1: What are the common methods of obtaining initial feasible solution in a transportation problem? (3 Marks) Nov./02 & (3 Marks) Nov./08-O.C.

Steps for Stage 2 (Optimality Test)

It can be applied to a transportation table if it satisfies the following conditions:

a) It contains exactly m+n-1 allocations, where m & n represent the no. of rows & columns of the table.

{***Note:*** (i) In case degeneracy occurs (i.e. allocations are less than m+n-1), infinitely small allocation(s) i.e. ε is/are introduced in *least cost & independent* cell to non-degenerate the solution. If least cost cell(s) is/are not independent, then cell(s) with next lower cost(s) is/are chosen.

(ii) Degeneracy can occur at initial basic feasible solution or at a later stage. ε may be removed once the purpose is over.

(iii) The quantity, ε is considered to be so small that if it is transferred to an occupied cell, it does not change the quantity of allocation, i.e. $x_{ij} - \varepsilon = x_{ij}$; $x_{ij} + \varepsilon = x_{ij}$; $\varepsilon - \varepsilon = 0$; $\varepsilon + \varepsilon = \varepsilon$; $x_{ij} \times \varepsilon = x_{ij}$}

b) These allocations are independent i.e. loop can not be performed by them.

{***Note:*** (i) Closed path or loop should have even no. of turns & is formed with an allocation on each corner, which in turn is a join of horizontal & vertical (not diagonal) lines. A loop may not involve all the allocations.

(ii) IBFS obtained by Northwest Corner rule, Least cost method or VAM are always in independent

positions though they may be m+n-1 or less than m+n-1 in no.}

The Stepping-Stone Method > **LEAST Important Topic**

1. Start with an arbitrary empty cell (a cell without allocation a.k.a. non-basic variables) and allocated +1 unit to this cell.
2. In order to maintain conditions of requirement & supply (a.k.a. rim constraints), 1 unit will be deducted/added to basic variables (i.e. cells having allocations).
3. Calculate Net change (a.k.a. opportunity cost or net evaluation) in transportation cost as a result of this perturbation.
4. Calculation Net evaluation for every empty cell.
5. If any cell evaluation is -ve, the cost can be reduced so that the solution under consideration can be improved i.e. it is not optimal. On the other hand if all net evaluation are +ve, the given solution is optimal one. If any net evaluation is Zero, the given solution is optimal but there exist other optimal solutions as well.
6. If solution is not optimal, the cell with largest –ve opportunity cost should be selected & allocate maximum units to this route (subject to rim constraints).
7. Iterate towards optimal solution, if further savings are possible.

Modified Distribution Method (MODI) -**Preferable**> **MOST Important Topic**

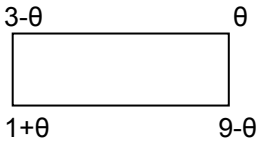
1. Form UV Matrix. It is formed
   (a) By adding a column to the RHS of transportation table labeled u_i & row in the bottom to it labeled v_j.
   (b) Assign any value arbitrarily to a row or column variable u_i or v_j.
   (c) Taking each cost cell $(u_i+v_j)=c_{ij}$, calculate individually all values of u_i & v_j

2. Calculate Net evaluation table, opportunity cost for each *unallocated cell* i.e. Δ_{ij} =i.e. $c_{ij}-(u_i+v_j)$ {Also, Opportunity cost = Actual cost – *Implicit* cost}

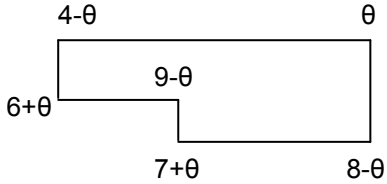
3. Check for signs of Δ_{ij}. If all Δ_{ij} are +ve, the given solution is optimal one. If any Δ_{ij} is Zero, the given solution is optimal but there exist other optimal solutions as well. If there are –ve Δ_{ij}, it means this solution is not optimal.

4. If solution is not optimal, the cell with largest –ve opportunity cost should be selected. Form a closed loop starting and ending at the selected worst –ve cell and reallocate the solution (maintaining rim constraints).

   E.g. If Reallocation is to done by transferring the max. amount to the ticked cell, the rule for obtaining max. amount (say θ_{max}) that can be transferred to the ticked cell is derived as:



$$\theta_{max} = \text{min of } \begin{Bmatrix} 3-\theta=0 \\ 9-\theta=0 \end{Bmatrix} = 3$$



$$\theta_{max} = \text{min of } \begin{Bmatrix} 4-\theta=0 \\ 8-\theta=0 \\ 9-\theta=0 \end{Bmatrix} = 4$$

5. Iterate towards optimal solution, if further savings are possible.

Question 2: What do you mean by Degeneracy in transportation problem? How this can be solved?
(4 Marks) May/10-N.C. & (4 Marks) Nov./09-O.C.

Question 3: How do you know whether an alternative solution exists for a transportation problem?

(4 Marks) Nov./09-N.C.

Question 4: In an unbalanced minimization transportation problem, with positive unit transport costs from 3 factories to 4 destinations, it is necessary to introduce a dummy destination to make it a balanced transportation problem. How will you find out if a given solution is optimal? (4 Marks) May/10-O.C.

[Ans.: (i) Yes (ii) Check for signs of Δ_{ij}. If all Δ_{ij} are +ve, the given solution is optimal one. If any Δ_{ij} is Zero, the given solution is optimal but there exist other optimal solutions as well. If there are –ve $\Delta_{ij,}$ it means this solution is not optimal.**]**

Question 5: Obtain the IBFS for the following using (i) North West Corner Rule (ii) LCM (iii) VAM.

	Warehouse-1	Warehouse-2	Warehouse-3	Supply
Factory-1	6	8	4	14
Factory-2	4	9	8	12
Factory-3	1	2	6	5
Demand	6	10	15	-

Question 6: Goods manufactured at 3 plants, A, B and C are required to be transported to sales outlets X, Y and Z. The unit costs of transporting the goods from the plants to the outlets are given below:

Sales outlets\Plants	A	B	C	Total Demand
X	3	9	6	20
Y	4	4	6	40
Z	8	3	5	60
Total supply	40	50	30	120

You are required to:
(i) Compute the initial allocation by North-West Corner Rule.
(ii) Compute the initial allocation by Vogel's approximation method and check whether it is optional.
(iii) State your analysis on the optionality of allocation under North-West corner Rule and Vogel's Approximation method. (10 Marks) May/08

[Ans.:

(i) Initial allocation under NW corner rule is:

Plant	Sales Outlet	Units	Initial Cost
A	X	20	60
A	Y	20	80
B	Y	20	80
B	Z	30	90
C	Z	30	150
			460

(ii) Initial allocation under VAM is:

Plant	Sales Outlet	Units	Initial Cost
A	X	20	60
A	Y	20	80
B	Z	50	150
C	Y	20	120
C	Z	10	100
			460

It is optimal solution

(iii) The solution under VAM is optimal with a zero in R_2C_2 which means that the cell C_2R_2 which means that the cell C_2R_2 can come into solution, which will be another optimal solution. Under NWC rule the initial allocation had C_2R_2 and the total cost was the same ₹460 as the total cost under optimal VAM solution. Thus, in this problem, both methods have yielded the optimal solution under the 1st allocation. If we do an optimality test for the solution, we will get a zero for Δ_{ij} in C_3R_2 indicating the other optimal solution which was obtained under VAM.**]**

Question 7 (Minimization-Unbalanced): A company has three plants located at A, B and C. The production of these plants is absorbed by four distribution centres located at X, Y, W and Z. The transportation cost per unit has been shown in small cells in the following table:

Distribution Centers / Factories	X	Y	W	Z	Supply (Units)
A	6	9	13	7	6000
B	6	10	11	5	6000

C	4	7	14	8	6000
Demand (Units)	4000	4000	4500	5000	18000 / 17500

Find the optimum solution of the transportation problem by applying Vogel's Approximation Method.
[Ans.: Total Cost ₹129500**]** (8 Marks) Nov./10-N.C.

Question 8 (Minimization-Unbalanced): A product is manufactured by four factories A, B, C & D. The unit production costs are ₹2, ₹3, ₹1 and ₹5 respectively. Their daily production capacities are 50, 70, 30 and 50 units respectively. These factories supply the product to four stores P, Q, R, S. The demand made by these stores are 25, 35, 105 and 20 units respectively. Unit transportation cost in rupees from each factory to each store is given in the following table:

	Stores			
Factories	**P**	**Q**	**R**	**S**
A	2	4	6	11
B	10	8	7	5
C	13	3	9	12
D	4	6	8	3

Determine the extent of deliveries from each of the factories to each of the stores so that the total cost (production and transportation together) is minimum. (10 Marks) May/02

[Ans.: Total Cost ₹1465**]**

Question 9 (Minimization-unbalanced-degeneracy): A company has four terminals U, V, W, and X. At the start of a particular day 10, 4, 6, and 5 trailers respectively are available at these terminals. During the previous night 13, 10, 6, and 6 trailers respectively were loaded at plants A, B, C, and D. The company dispatcher has come up with the costs between the terminals and plants as follows:

		Plants			
		A	**B**	**C**	**D**
Terminals	U	20	36	10	28
	V	40	20	45	20
	W	75	35	45	50
	X	30	35	40	25

Find the allocation of loaded trailers from plants to terminals in order to minimize transportation cost.
(10 Marks) May/97

[Ans.:

Terminal	Plant	Cost (₹)
U	A	80
U	C	60
V	B	60
V	D	20
W	B	210
X	D	125
	Transportation Cost	555**]**

Question 10 (Minimization-balanced-degeneracy): The cost per unit of transporting goods from factories X, Y, Z to destinations A, B and C, and the quantities demanded and supplied are tabulated below. As the company is working out the optimum logistics, the Govt. has announced a fall in oil prices. The revised unit costs are exactly half the costs given in the table. You are required to evaluate the minimum transportation cost.

Factories\Destinations	A	B	C	Supply
X	15	9	6	10
Y	21	12	6	10
Z	6	18	9	10
Demand	10	10	10	30

[Ans.: Min. transportation cost is ₹105**]** (6 Marks) June/09-N.C.

Question 11 (Minimization-balanced-degeneracy): Find optimal solution for the following problem

	Warehouse-1	Warehouse-2	Warehouse-3	Supply
Factory-1	50	30	220	1
Factory-2	30	45	170	3
Factory-3	250	200	50	4
Demand	4	2	2	

[Ans.: Optimal Cost is ₹640**]**

Question 12 (Degeneracy - ε is introduced at next least cost cell): The initial allocation of a transportation problem, alongwith the unit cost of transportation from each origin to destination is given below. You are required to arrive at the minimum transportation cost by the Vogel's Approximation method and check for optimality.
(Hint : Candidates may consider $u_1 = 0$ at Row 1 for initial cell evaluation)

					Requirement
11	2 [8]	8	6 [6]	2 [4]	18
9 [10]	9	12	9	6	10
7	6	3 [8]	7	7	8
9 [2]	3	5	6 [2]	11	4
Availability					
12	8	8	8	4	40

[Ans,: Optimal Cost = ₹204**]** (6 Marks) May/07
[Hint for solution: R_4C_2 has the least cost (cost = 3), but this is not independent. The next least cost cell R_4C_3 (cost = 5) is independent, hence we will select it to resolve the degeneracy.**]**

Question 13 (Minimization-unbalanced-degeneracy): A company has factories at A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 160, 150 and 190 units respectively. Monthly warehouse requirements are 80, 90, 110 and 160 units respectively. Unit shipping costs (in rupees) are as follows:

	To			
From	**D**	**E**	**F**	**G**
A	42	48	38	37
B	40	49	52	51
C	39	38	40	43

Determine the optimum distribution for this company to minimize shipping costs.

[Ans.: Minimal Cost is ₹17050**]**

Question 14: Find optimal solution for the following transportation cost problem.

	Go-down 1	Go-down 2	Go-down 3	Go-down 4	Go-down 5	Go-down 6	Supply
Factory-1	4	6	9	2	7	8	10
Factory-2	3	5	4	8	10	0	12
Factory-3	2	6	9	8	4	13	4
Factory-4	4	4	5	9	3	6	18
Factory-5	9	8	7	3	2	14	20
Demand	8	8	16	3	8	21	-

[Ans.: ₹242**]**
***NOTE*: SIMILAR QUESTION ↑ IS WRONGLY SOLVED BY STUDY MATERIAL OF ICAI**

Question 15 (Prohibited routes): Solve the following transportation problem

	Go-down 1	Go-down 2	Go-down 3	Go-down 4	Go-down 5	Go-down 6	Stock availability
Factory 1	7	5	7	7	5	3	60
Factory 2	9	11	6	11	*	5	20
Factory 3	11	10	6	2	2	8	90
Factory 4	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	

Note: It is not possible to transport any quantity from factory 2 to go-down 5. State whether the solution derived by you is unique. (Nov./1989)

[Ans.: Total Cost ₹1120**]**

Question 16 (Multiple Optimal Solutions): A Compressed Natural Gas (CNG) company has three plants producing gas and four outlets. The cost of transporting gas from different production plants to the outlets, production capacity of each plant & requirement at different outlet is shown in the following cost-matrix table.

	Outlets				
Plants	**A**	**B**	**C**	**D**	**Capacity of Production**
X	4	6	8	6	700
Y	3	5	2	5	400
Z	3	9	6	5	600
Requirement	400	450	350	500	1700

Determine a transportation schedule so that the cost is minimized.
The cost in the cost-matrix is given in thousand of rupees. (10 Marks) Nov./01

[Ans.: Minimum costs ₹7350 thousands**]**

Question 17 (Maximization-unbalanced-degeneracy-multiple solution): A particular product is manufactured in factories A, B, C and D; and is sold at centres 1, 2, and 3. The cost in ₹ of product per unit and capacity in kgms per unit time of each plant is given below:

Factory	Cost (₹) per unit	Capacity (kgms) per unit
A	12	100
B	15	20
C	11	60
D	13	80

The sale price in ₹per unit and the demand in kgms per unit time are as follows:

Sales Centre	Sale Price (₹) per unit	Demand (kgms) per unit
1	15	120
2	14	140
3	16	60

Find the optimal sale distribution. (10 Marks) Nov./97

[Ans.:

From Factory	To Sales Centre	Quantity	Total Profit
A	1	100	300
B	2	20	-20
C	3	60	300
D	1	20	40
D	2	60	60
Dummy	2	60	0
			680**]**

Question 18 (Investment decision): XYZ and co. has provided the following data seeking your advice on optimum investment strategy:

Beginning of the year	Net Return Data (in paise) of Selected Investments				Amount available (Lacs)
	P	Q	R	S	
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Maximum Investment (lacs)	40	50	60	60	-

The following additional information are also provided:

- P, Q, R, and S represent the selected investments.
- The company decided to have four years investment plan.
- The policy of the company is that amount invested in any year will remain so until the end of the fourth year.
- The values (paise) in the table represent net return on investment of one Rupee till the end of the planning horizon (for example, a Rupee invested in Investment P at the beginning of year 1 will grow to ₹1.95 by the end of the fourth year, yielding a return of 95 paise).

Using the above, determine the optimum investment strategy. (10 Marks) Nov/96

[Ans.: Total investment ₹13000000**]**

Question 19 (Maximization-unbalanced): ABC Enterprises is having three plants manufacturing dry-cells, located at different locations. Production cost differs from plant to plant. There are five sales offices of the company located in different regions of the country. The sales prices can differ from region to region. The shipping cost from each plant to each sales office and other data are given by following table:

Production Data Table		
Production cost per unit	**Max. capacity in no. of units**	**Plant no.**
20	150	1
22	200	2
18	125	3

Shipping Cost and Demand & Sales Prices table					
Shipping Costs					
	Sales Office 1	**Sales Office 2**	**Sales Office 3**	**Sales Office 4**	**Sales Office 5**
Plant 1	1	1	5	9	4
Plant 2	9	7	8	3	6
Plant 3	4	5	3	2	7
Demand & Sales Prices					
	Sales Office 1	**Sales Office 2**	**Sales Office 3**	**Sales Office 4**	**Sales Office 5**
Demand	80	100	75	45	125
Sales Price	30	32	31	34	29

Find the production and distribution schedule most profitable to the company. (10 Marks) Nov/98

[Ans.:

Plant	Sales Office	Units	Total Profit (₹)
1	1	50	450
1	2	100	1100
2	4	25	225
2	5	125	125
2	Dummy	50	0
3	1	30	240
3	3	75	750
3	4	20	280
	Total		3170**]**

Question 20 (Maximization-unbalanced): A company has three factories and four customers. The company furnishes the following schedule of profit per unit on transportation of its goods to the customers in rupees:

	Customers				
Factory	**A**	**B**	**C**	**D**	**Supply**
P	40	25	22	33	100
Q	44	35	30	30	30
R	38	38	28	30	70
Demand	40	20	60	30	

You are required to solve the transportation problem to maximize the profit and determine the resultant of optimal profit. (8 Marks) May/03

[Ans.: Maximum Profit ₹5130**]**

Question 21 (Maximization-unbalanced-degeneracy): A leading firm has three auditors. Each auditor can work up to 160 hours during the next month, during which time three projects must be completed. Project 1 will 130 hours, project 2 will take 140 hours, and project 3 will take 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given in Table 1:

Table1

	Project		
Auditor	**1**	**2**	**3**
	₹	₹	₹
1	1200	1500	1900
2	1400	1300	1200
3	1600	1400	1500

Formulate this as a transportation problem and find the optimal solution. Also find out the maximum total billings during the next month. (10 Marks) May/95

[Ans.:

Auditor	Project	Billing Amount (₹)
1	3	304000
2	2	143000
3	1	208000
3	2	42000
	Total billings	697000**]**

Miscellaneous Questions:

Question 22 (Overtime Production): A company has factories at A, B, and C, which supply warehouses at D, E, F and G. The factories capacities are 230, 280, and 180 respectively for regular production. If overtime production is utilised, the capacities can be increased to 300, 360 and 190 respectively. Increment unit overtime costs are ₹5, ₹4, and ₹6 respectively. The current warehouse requirements are 165, 175, 205 and 165 respectively. Unit shipping costs in rupees between the factories and the warehouses are

To From	D	E	F	G
A	6	7	8	10
B	4	10	7	6
C	3	22	2	11

Determine the optimum distribution for the company to minimize costs.

[Ans.:

From	To	Quantity
A	D	40
A	E	175
A	F	15
B	D	115
B	G	165
C	F	180
A_1	Dummy	70
B_1	D	10
B_1	Dummy	70
C_1	F	10**]**

Question 23 [Redundancy cost]: A company has seven manufacturing units situated in different parts of the country. Due to recession it is proposed to close four of these and to concentrate production in the remaining three units. Production in these units will actually be increased from present level and would require an increase in the personnel employed in them. Personnel at the closed units expressed their desire from moving to any one of the remaining units and the company is willing to provide them removal (transfer) costs. The retraining expenses would have to be incurred as the technology in these units are different. Not all existing personnel can be absorbed by transfer and a number of redundancies will arise, cost of redundancy is given as a general figure at each unit closed.

		A	**B**	**C**	**D**
No. Employed		200	400	300	200
		(These units A, B, C and D are to be closed)			
Retraining costs in ₹ '000/person		A	B	C	D
Transfer to	E	0.5	0.4	0.6	0.3
	F	0.6	0.4	0.6	0.3
	G	0.5	0.3	0.7	0.3
Removal cost in ₹'000/person		A	B	C	D
Transfer to	E	2.5	3.6	3.4	4.7
	F	2.4	4.6	3.4	1.7
	G	2.5	2.7	3.3	2.7
Redundancy payments in ₹'000/person		A	B	C	D
		6.0	5.0	6.0	7.0
Additional personnel required at Units		E	F	G	
Required		350	450	200	

(i) Obtain a solution to the problem of the cheapest means to transfer personnel from units closed to those units which will be expanded.
(ii) State with reason whether or not the solution obtained is optimal and unique.
(iii) State the costs of the initial and final solution. [ICWA /1990]

[Ans.: (i)

From	To	Quantity	Total Cost (₹'000s)
A	E	200	600
B	E	100	400
B	G	200	600
B	Dummy	100	500
C	E	50	200
C	F	250	1000
D	F	200	400
			3700

(ii) Since there exist Zero Δ_{ij} hence, solution is optimal but not unique. (iii) Costs are same i.e.₹3700000**]**

Trans-shipment Model (LEAST IMPORTANT)

The trans-shipment model recognizes that it may be cheaper to ship through intermediate or transient nodes before reaching the final destination. This concept is more general than that of the regular transportation model, where direct shipments only are allowed between a source and a destination.
In the tras-shipment model, as each source or destination is a potential point of supply as well as demand, total supply, say of B units is added to the actual supply of each source as well as to the actual demand at each destination. Also the 'demand' at each source & 'supply' at each destination is set equal to B. Therefore, we may assume the supply & demand of each location to be fictitious one. These quantities (B) may be regarded as buffer stocks & each of these buffer stocks should at least be equal to the total supply/demand in the given problem.

Question 24: A firm has two factories X and Y and three retail stores A, B and C. The number of units of a product available at factories X and Y are 200 and 300 respectively, while demanded at retail stores are 100, 150 and 250 respectively. Rather than shipping directly from sources to destinations, it is decided to investigate the possibility of trans-shipment. Find the optimal shipping schedule. The transportation costs in rupees per unit are given below.

		Factory		Retail Store		
		X	Y	A	B	C
Factory	X	0	6	7	8	9
	Y	6	0	5	4	3
Retail Store	A	7	2	0	5	1
	B	1	5	1	0	4
	C	8	9	7	6	0

[Ans.: Factory X supplies 100 units each to retail stores A and B. Factory Y supplies 50 units to retail store B and 250 units to C.**]**

Question 24: Table 1 represents the supply from the plants, the requirement at the distribution centres and the unit transportation costs.

Table 1 – Distribution Centres					
		A	B	C	Supply
Plants	1	11	13	25	150
	2	13	15	35	300
Requirement		150	150	150	

When each plant is also considered a destination and each distribution centre is also considered as origin, some additional cost data are necessary, which are given in the tables below:

Table 2 - To Plants				Table 3 - To dist. centres					Table 4 - To plants			
		1	2			A	B	C			1	2
From Plant	1	0	75	From Dist. Centre	A	0	33	11	From Dist. Centre	A	13	25
	2	11	0		B	11	0	13		B	35	13
					C	75	13	0		C	55	65

Find the optimal shipping schedule for the trans-shipment problem.

[Ans.: Plant 2 supplies 450 units each to Distribution Center A & B; Plant 1 transports 150 units to distribution centre A. Distribution centre A sends 150 units to C; Total cost of transshipment ₹7,500**]**

12

The Assignment Problem

The assignment problem like transportation is another special case of LPP. In general, it is concerned with one to one basis when ***n*** jobs are to be assigned to ***n*** facilities with a view to optimizing the resource required. The emphasis in an assignment problem is on determining how assignment should be made in order that the total cost involved is minimized or alternatively, total value is maximized when pay-offs are given in terms of say profits.

Steps for solving assignment problem (Using Hungarian Assignment Method-HAM):

1) If problem is not balanced i.e. m≠n, introduce dummy row/column to balance it.
   Where m= no. of rows & n= no. of columns.
2) In case of problems of maximization, Convert profit matrix into loss matrix. This is done by subtracting each entry in the table from largest no. in the table.
   [Important Note: Though we can also use step 2 before step 1(i.e. we can also convert profit matrix in loss matrix before balancing the problem), but, it is recommended to apply the steps in above defined order only]
3) Assign M or ∞ to prohibited assignments, etc.
4) Subtract the minimum element of each row from all the elements in that row.
   [Note: Step 4 is to calculate Men/Machine/Facility/etc. opportunity costs with regards to the respective jobs/tasks/etc. (i.e. penalty for assigning to a facility, machine, etc. in spite of least cost assignment.)]
5) Subtract the minimum element of each column from all the elements in that column. The resulting matrix is starting matrix.
   [Note: Step 5 is to calculate total opportunity cost table.]
6) Draw the *min.* no. of lines (both horizontal & vertical) to cover all zeros.
7) Now check that whether *n* i.e. order of matrix & no. of lines drawn in Step 6 are same or not. If yes, go to step 10, otherwise go to step 8 (for improving the solution).
   [Note: Order of matrix means no. of rows/column in a matrix]
8) Find the smallest uncovered cost element (by the lines drawn in step 7). Subtract this element from all such (uncovered) elements & add it to the elements with two lines through it (i.e. intersection of horizontal & vertical lines). Do not alter the elements through which one line passes.
   [Note: Step 8 gives us revised opportunity cost table]
9) Repeat Steps 6-8 until we get minimum no. of lines equal to *n.*
10) Find a row with only one zero in it, & make an assignment (by encircling it) to this zero. Draw a vertical line through the *column* containing this zero. This eliminates any confusion of making any further assignment in that column. Repeat this step until no row with single zero is left.
11) Find a column with only one zero in it, & make an assignment to this zero. Draw a horizontal line through the *row* containing this zero. This eliminates any confusion of making any further assignment in that row. Repeat this step until no column with single zero is left.
12) If at this stage, there are still zeroes which are not assigned or crossed, then go to step 13 otherwise the solution is optimal.
13) If any row/column consists more than one zero, make an assignment to any zero arbitrarily & draw 2

lines i.e. one horizontal & one vertical line through the row & column containing this zero.
14) Repeat steps 10) to 13) until no further movement is possible. The resulting solution is optimal.
15) Determine the total cost with reference to the original cost table.

Question 1: In an assignment problem to assign jobs to men to minimise the time taken, suppose that one man does not know how to do a particular job, how will you eliminate this allocation from the solution?
(4 Marks) Nov./09-N.C.

Question 2 (Minimization balanced): A Marketing Manager has 4 subordinates and 4 tasks. The subordinates differ in efficiency. The tasks also differ in their intrinsic difficulty. His estimates of the time each subordinate would take to perform each task is given in the matrix below. How should the task be allocated one to one man so that the total man-hours are minimized? (7 Marks) Nov./04

	I	II	III	IV
1	16	52	34	22
2	26	56	8	52
3	76	38	36	30
4	38	52	48	20

[Ans.: Minimum time taken = 82 hours**]**

Question 3: Find the optimal solution for the assignment problem with the following cost matrix:

Marketing	Division			
Executive	N	E	W	S
A	14	20	11	19
B	12	10	15	9
C	16	19	18	15
D	17	13	15	14

[Ans.: Total minimum cost ₹49**]** (10 Marks) Nov./03

Question 4 (Maximization-balanced): A manager has 4 subordinates and 4 tasks. The subordinates differ in efficiency. His estimate of the production each would do is given in the table. How the task should be allocated one to one man, so that total production is maximized.

	Subordinates			
Task	I	II	III	IV
1	8	26	17	11
2	13	28	4	26
3	38	19	18	15
4	19	26	24	10

[Ans.: Optimal Production is 114 units**]**

Question 5 (Maximization-balanced): An organization producing 4 different products viz. A, B, C, and D having 4 operators viz. P, Q, R and S, who are capable of producing any of the four products, works effectively 7 hours a day. The time (in minutes) required for each operator for producing each of the product are given in the cells of the following matrix along with profit (₹per unit):

	Products			
Operators	**A**	**B**	**C**	**D**
P	6	10	14	12
Q	7	5	3	4
R	6	7	10	10
S	20	10	15	15
Profit (₹/unit)	3	2	4	1

Find the optimal assignment of operators to products which will maximize the profit.

[Ans.:

Operator	Product	Profit
P	A	210
Q	C	560
R	B	120
S	D	28**]**

(10 Marks), May/96 & (10 Marks), Nov./97 - Adapted

Question 6 (Minimization-unbalanced): A has one surplus truck in each cities A, B, C, D & E and one deficit truck in each of the cities 1, 2, 3, 4, 5 & 6. The distance between the cities in kilometers is shown in the matrix below.

Cities	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	3	10

Find the assignment of trucks from the cities in surplus to cities in deficit so that the total distance covered by vehicles in minimum.

[Ans.: Optimum Distance is 34 kms**]**

Question 7 (Multiple optimal solutions): Solve the minimal assignment problem whose effectiveness matrix is

	Men			
Jobs	**1**	**2**	**3**	**4**
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

[Ans.: Optimal cost is ₹20/-**]**

Question 8 (Multiple Optimal Solutions): The cost matrix giving selling costs per unit of a product by salesman A, B, C and D in Regions R_1, R_2, R_3 and R_4 is given below:

(i) Assign one salesman to one region to minimize selling cost.

(ii) If the selling price of the product is ₹200 per unit and variable cost excluding the selling cost given in the table is ₹100 per unit, find the assignment that would maximize the contribution.

(iii) What other conclusion can you make from the above?

	A	B	C	D
R_1	4	12	16	8
R_2	20	28	32	24
R_3	36	44	48	40
R_4	52	60	64	56

(8 Marks) Nov./08-N.C.

[Ans.: R_1->A, R_2->B, R_3 ->C and R_4->D with minimum selling cost as ₹136; (ii) R_1->A, R_2->B, R_3 ->C and R_4->D with maximum contribution as ₹264**]**

Question 9 (Prohibited routes): A BPO Co. is taking bids for 4 routes in the city to ply pick-up and drop cabs. Four companies have made bids as detailed below: Bids for Routes (₹) :

Company/Routes	R1	R2	R3	R4
C1	4,000	5,000	---	---
C2	---	4,000	---	4,000
C3	3,000	---	2,000	---
C4	---	---	4,000	5,000

Each bidder can be assigned only one route. Determine the minimum cost.

[Ans.: ₹15000/-**]** (10 Marks) Nov./95 & (6 Marks) Nov./06

Question 10 (Prohibited routes): WELLDONE Company has taken the third floor of a multi-storied building for rent with a view to locate one of their zonal offices. There are five main rooms in this floor to be assigned to five managers. Each room has its own advantages and disadvantages. Some have windows; some are closer to the washrooms or to the canteen or secretarial pool. The rooms are of all different sizes and shapes. Each of the five managers was asked to rank their room preferences amongst the rooms 301, 302, 303, 304 and 305. Their preferences were recorded in a table as indicated below:

MANAGER				
M_1	M_2	M_3	M_4	M_5
302	302	303	302	301
303	304	301	305	302
304	305	304	304	304
*	301	305	303	*
*	*	302	*	*

Most of the managers did not list all the five rooms since they were not satisfied with some of these rooms and they have left off these from the list. Assuming that their preferences can be quantified by numbers, find out as to which manager should be assigned to which room so that their total preference ranking is a minimum. (May/1988 & Nov./1990)

[Ans.: $M_1 \rightarrow 302$, $M_2 \rightarrow 304$, $M_3 \rightarrow 303$, $M_4 \rightarrow 305$ and $M_5 \rightarrow 301$ with total minimum ranking as 7**]**

Question 11 (Unbalanced-Prohibited routes): A factory is going to modify a plant layout to install four new machines M_1, M_2, M_3 and M_4. There are 5 vacant places J, K, L, M and N available. Because of limited space machine M_2 cannot be placed at L and M_3 cannot be placed at J. The cost of locating machine to place in Rupees is shown below:

(₹)

	J	K	L	M	N
M_1	18	22	30	20	22
M_2	24	18	-	20	18
M_3	-	22	28	22	14
M_4	28	16	24	14	16

Determine the optimal assignment schedule in such a manner that the total costs are kept at a minimum.

[Ans.: $M_1 \rightarrow J$, $M_2 \rightarrow K$, $M_3 \rightarrow N$, & $M_4 \rightarrow M$ with total minimum cost of ₹64**]** (7 Marks) June/09-O.C.

Question 12 (Maximization-Prohibited route): Imagine yourself to be the Executive Director of a 5-star Hotel which has four banquet halls that can be used for all functions including weddings. The halls were all about the same size and the facilities in each hall differed. During a heavy marriage season, 4 parties approached you to reserve a hall for the marriage to be celebrated on the same day. These marriage parties were told that the first choice among these 4 halls would cost ₹10000 for the day. They were also required to indicate the second, third and fourth preferences and the price that they would be willing to pay. Marriage party A & D indicated that they won't be interested in Halls 3 & 4. Other particulars are given in the following table:

	Revenue/Hall			
	Halls			
Marriage Party	1	2	3	4
A	10000	9000	X	X
B	8000	10000	8000	5000
C	7000	10000	6000	8000
D	10000	8000	X	X

Where X indicates that the party does not want that hall.

Decide on an allocation that will maximize the revenue to your hotel. (10 Marks) May/95

[Ans.: Marriage Party Revenue (₹)

A	Hall 2	9000
B	Hall 3	8000
C	Hall 4	8000
D	Hall 1	10000**]**

Question 13 (Airline scheduling): An airline operates seven days a week has time-table as shown below. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in smaller layover. For each pair also mention the town where the crew should be based.

	Delhi	Jaipur		Jaipur	Delhi
Flight no.	**Departure**	**Arrival**	**Flight no.**	**Departure**	**Arrival**
1	7.00	8.00	101	8.00	9.15
2	8.00	9.00	102	8.30	9.45
3	13.30	14.30	103	12.00	13.15
4	18.30	19.30	104	17.30	18.45

[Ans.: Minimum Layover time is 52.5 hours**]** (10 Marks) May/97-Adapted & (10 Marks) May/00-Adapted

Question 14 (Data entry job allocation): A firm employs typists for piecemeal work on an hourly basis. There are five typists available and their charges and speeds are different. According to an earlier understanding, only one job is given to one typist and the typist is paid for full hours even if he works for a fraction of an hour. Find the least cost allocation for the following data:

Typist	Rate/Hour	Pages/Hour	Job	No. of Pages
A	₹5	12	P	199
B	₹6	14	Q	175
C	₹3	8	R	145
D	₹4	10	S	298
E	₹4	11	T	178

[Ans.: (10 Marks) Nov./96

Typist	**Job**	**Cost(₹)**
A	T	75
B	R	66
C	Q	66
D	P	80
E	S	112
		399**]**

Question 15 (Traveling salesmen-LEAST IMPORTANT): A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and return to his starting point. The travelling cost for each cty from a particular city is given below:

		To City				
		A	B	C	D	E
From City	A	X	4	7	3	4
	B	4	X	6	3	4
	C	7	6	X	7	5
	D	3	3	7	X	7
	E	4	4	5	7	X

What is the sequence of visit of the salesman, so that the cost is minimum?

[Hint: Route A to B, B to D, D to A shall not be taken because D is not allowed to follow A, till C & E are

processed. So we will follow next best solution i.e. A-E-C-B-D-A**]** (I.C.W.A. Final, June/97-Adapted)

Question 16 (Replacement decisions): Average time taken by an operator on a specific machine is tabulated below. The management is considering replacing one of the old machines by a new one and the estimated time for operation by each operator on the new machine is also indicated.

	Machines						
Operators	**1**	**2**	**3**	**4**	**5**	**6**	**New**
A	10	12	8	10	8	12	11
B	9	10	8	7	8	9	10
C	8	7	8	8	8	6	8
D	12	13	14	14	15	14	11
E	9	9	9	8	8	10	9
F	7	8	9	9	9	8	8

(a) Find out an allocation of operators to the old machines to achieve a minimum operation time.
(b) Reset the problem with the new machine and find out the allocation of the operators to each machine and comment on whether it is advantageous to replace an old machine to achieve a reduction in operating time only.
(c) How will the operators be reallocated to the machines after replacement?

[Ans.: Old Allocation A-3,B-4,C-6,D-1,E-5,F-2 with time = 49 New Allocation A-3,B-4,C-6,D-New,E-5,F-1 with time =47**]**

Question 17: A company has four zones open and four marketing managers available for assignment. The zones are not equal in sales potentials. It is estimated that ***a typical marketing manager operating in*** each ***zone*** would bring in the following Annual sales:

Zones	₹
East	2,40,000
West	1,92,000
North	1,44,000
South	1,20,000

The four marketing managers are also different in ability. It is estimated that working under the same conditions, their yearly sales would be proportionately as under:

Manager M : 8
Manager N : 7
Manager O : 5
Manager P : 4

Required:
If the criterion is maximum expected total sales, find the optimum assignment and the maximum sales.

[Ans.: Assignment	**Sales (₹)**
M – East	80,000
N – West	56,000
O – North	30,000
P – South	20,000
	1,86,000]

(11Marks) Nov./07 & (10 Marks) May/98-Adapted

{***Caution:*** *Please make sure that you do the above question in accordance with answer of similar questions solved by me or provided by C.A. Institute's Suggested Answers –* ***Please abstain from using any other method suggested by any other teacher of Delhi***}

Question 18 (Liquidity management): X holds stock of different companies. For a certain problem he is compelled to sell off four of his holdings. Since he wants the money over the next five months, he sells not more than one stock in any month. He has estimated the sale proceeds in each of the next five months as follows:

(₹'000)

	June	July	Aug	Sep	Oct
S1	13	16	14	19	17
S2	18	20	13	18	12
S3	17	15	10	22	18
S4	19	18	14	21	15

Find what will be the optimum plan of X and how much money can he realize by sales?

[Ans.: Optimum sale proceeds are ₹78000**]**

Question 19 (Market research and assignment): The market research team of the Look Forward Company requires some household data from four different cities. The team has to perform this job in two days- the next Saturday and Sunday. It plans to spend half a-day in each of the cities. The relevant data are given here:

Day and Time	Probability of a Household Contact			
	City 1	City 2	City 3	City 4
Saturday Morning	0.32	0.85	0.16	0.64
Saturday Evening	0.60	0.56	0.95	0.80
Sunday Morning	0.70	0.35	0.40	0.62
Sunday Evening	0.10	0.72	0.64	0.90
Number of households expected to interview	150	100	200	200

How should the team plan its visit to the four cities so that the expected response may be maximized? State this expected response.

[Ans.: Expected response is 560**]**

Question 20: A hospital has to pay nurses for 40 hours a week. One nurse is assigned to one patient. The cost per hour for each of the nurses is given below:
(i) Find the nurse-patient combination to minimize cost to the hospital.
(ii) How much does each nurse earn per week?

Nurse\Patient	W	X	Y
K	10	10	30
L	30	10	20
M	20	30	20

Suppose that a new patient Z is admitted, and that a new nurse N is appointed. The new patient is charged ₹40 per hour by each of the existing nurses. The new nurse charges ₹50 per hour irrespective of the patient.
(iii) What would be your revised calculations?
(iv) Comment on the new solution. (8 Marks) May'10-O.C.

[Ans.: (i) K→W; L→X; M→Y (ii) 400, 400, 800 (iii) K→W; L→X; M→Y; N→Z & 400, 400, 800, 2000**]**

Question 21: A city corporation has decided to carry out road repairs on main four arteries of the city. The government has agreed to make a special grant of ₹50 lakhs towards the cost with a condition that the repairs must be done at the lowest cost and quickest time. If conditions warrant, then a supplementary token grant will also be considered favorably. The corporation has floated tenders and 5 contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.

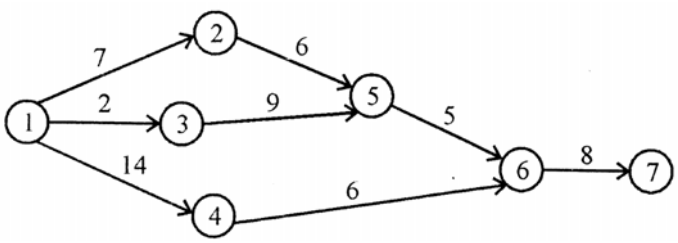
Cost of Repairs (₹Lakhs)

Contractors/Road	R1	R2	R3	R4
C1	9	14	19	15
C2	7	17	20	19
C3	9	18	21	18
C4	10	12	18	19
C5	10	15	21	16

(i) Find the best way of assigning the repair work to the contractors and the costs.
(ii) If it is necessary to seek supplementary grants, then what should be amount sought?
(iii) Which of the five contractors will be unsuccessful in his bid?
(iv) If C1 unable to accept any work, find best assignment.

[Ans.: C1-R3, C2-R1, C3-Dummy, C4-R2, C5-R4; Grant of ₹4 Lakhs; C3**]**

13



Network Analysis – PERT/CPM

PERT – Program Evaluation & Review Technique – It is generally used for those projects where time required to complete various activities are not known as *a priori*. It is probabilistic model & is primarily concerned for evaluation of time. It is event oriented.

CPM – Critical Path Analysis – It is a commonly used for those projects which are repetitive in nature & where one has prior experience of handling similar projects. It is a deterministic model & places emphasis on time & cost for activities of a project.

- A project can be defined as a set of large number of activities or jobs (with each activity consuming time & resources) that are performed in a certain sequence determined.
- A network is a graphical representation of a project, depicting the flow as well as the sequence of well-defined activities & events.
- An activity (**Also known as task & job**) is any portion of a project which consumes time or resources and has definable beginning & ending.
- Event (**Also known as node & connector**) is the beginning & ending points of an activity or a group of activities.

Steps for drawing CPM/PERT network: (4 Marks) Nov./10-N.C.

1. Analyze & break up of the entire project into smaller systems i.e. specific activities and/or events.
2. Determine the interdependence & sequence of those activities.
3. Draw a network diagram.
4. Estimate the completion time, cost, etc. for each activity.
5. Identify the critical path (longest path through the network).
6. Update the CPM/PERT diagram as the project progresses.

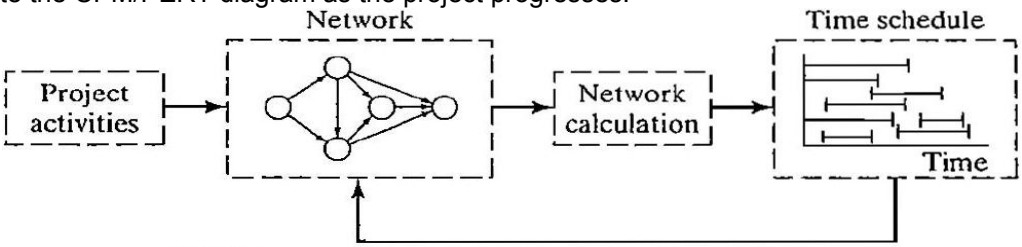


FIGURE Phases for project planning with CPM-PERT

Network Representation:

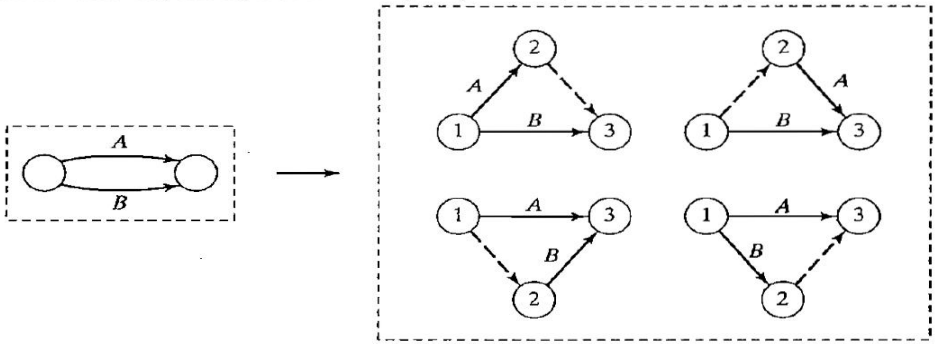
Each activity of the project is represented by arrow pointing in direction of progress of project. The events of the network establish the precedence relationship among different activities.
Three rules are available for constructing the network.

Rule 1. *Each activity is represented by one & only one, arrow.*

Rule 2. *Each activity must be identified by two distinct events* ***&*** *No two or more activities can have the same tail and head events.*
Following figure shows how a dummy activity can be used to represent two concurrent activities, A & B. Dummy activity is a hypothetical activity which takes no resource or time to complete. It is represented by broken arrowed line & is used for either distinguishing activities having common starting & finishing events or to identify & maintain proper precedence relationship between activities that are not connected by events.

Space for lecture notes

FIGURE Use of dummy activity to produce unique representation of concurrent activities



Inserting dummy activity in one four ways in the figure, we maintain the concurrence of A & B, and provide unique end events for the two activities (to satisfy Rule 2).

Rule 3. *To maintain correct precedence relationship, the following questions must be answered as each activity is added to the network:*

(a) What activities must be immediately precede the current activity?
(b) What activities must follow the current activity?
(c) What activities must occur concurrently with the current activity?

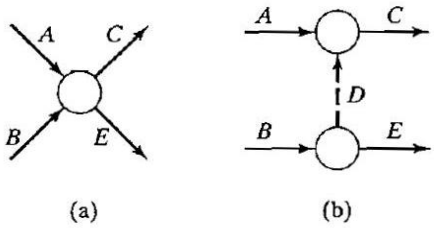


FIGURE Use of dummy activity to ensure correct precedence relationship

The answers to these questions may require the use of dummy activities to ensure correct precedence's among the activities. For example, consider the following segment of a project:

1. Activity C starts immediately after A and B has been completed.
2. Activity E starts only after B has been completed.

Part (a) of the figure above, shows the incorrect representation of the precedence relationship because it requires both A & B to be completed before E can start. In part (b) the use of dummy rectifies situation.

Question 1: What do you mean by a dummy activity? Why it is used in networking? (4 Marks) May/08
[Ans.: Dummy activity is a hypothetical activity which takes no resource or time to complete. It is represented by broken arrowed line and is inserted in the network to clarify an activity pattern under the following situations.

(a) For distinguishing activities having common starting & finishing events
(b) To identify & maintain proper precedence relationship between activities that are not connected by events.
(c) To bring all "loose ends" to a single initial and single terminal event.**]**

Question 2: Explain the following in the context of a network:

(i) Critical path
(ii) Dummy activity (4 Marks) June/09-N.C.

Question 3 (Application of Dummy): Depict the following dependency relationships by means of network diagrams.(The Alphabets stands for activities)

1. A and B control F; B and C control G.

2. A controls F, G and H; B controls G and H with H controlled by C.
3. A controls F and G; B controls G while C controls G and H.
4. A controls F and G; B and C control G with H depending upon C.
5. F and G are controlled by A; G and H are controlled by B with H controlled by B and C.
6. A and B control F; B controls G while C controls G and H.

Question 4: Develop a network based on the following information;

Activity	Immediate predecessors
A	-
B	-
C	A
D	B
E	C,D
F	D
G	E
H	F

Question 5 [Difficult network-Try it at only after proper practice]: Construct the project network comprised of activities A to L with the following precedence relationships:

(a) A,B and C, the first activities of the project can be executed concurrently
(b) A & B precede D
(c) B precedes E,F,H
(d) F and C precede G
(e) E and H precede I & J
(f) C,D,F and J precede K
(g) K precede L
(h) I, G, and L are terminal activities of the project.

[Hint: Network diagram of this question is same as that of Question 16**]**

Question 6 [Difficult network-Try it at only after proper practice]: Construct the project network comprised of activities A to P that satisfies the following precedence relationships:

(a) A,B and C, the first activities of the project can be executed concurrently
(b) D,E and F follow A
(c) I and G follow both B and D
(d) H follows both C & G
(e) K and L follow I
(f) J succeeds both E and H
(g) M and N succeed F, but cannot start until both E and H are completed.
(h) O succeeds both M and I
(i) P succeeds J,L and O
(j) K,N and P are the terminal activities of the project.

Question 7: A publisher has a contract with an author to publish a textbook. The simplified (activities) associated with the production of the textbook are given below. The author is required to submit to the publisher a hard copy and a computer file of the manuscript. Develop the associated network for the project.

Activity	Predecessor(s)	Duration (weeks)
A: Manuscript proofreading by editor	-	3
B: Sample pages preparation	-	2
C: Book cover design	-	4
D: Artwork preparation	-	3
E: Author's approval of edited manuscript and sample pages	A,B	2
F: Book formatting	E	4
G: Author' review of formatted pages	F	2
H: Author's review of artwork	D	1
I: Production of printing plates	G,H	2
J: Book production and binding	C,I	4

Question 8: A project consists of a series of tasks labeled A, B,……..., H, I with the following relationships (W < X,Y means X and Y cannot start until W is completed; X,Y < W means W cannot start until both X and Y are completed). With this notation construct the network diagram having the following constraints:

A < D,E; B,D < F; C < G; B < H; F,G < I

Question 9 (Concurrent Activities-LEAST IMPORTANT-Do after proper practice of network diagrams): The footing of a building can be completed in four consecutive sections. The activities for each section include (1) digging, (2) placing steel, and (3) pouring concrete. The digging of one section cannot start until that of the preceding section has been completed. The same restriction applies placing steel & pouring concrete. Develop the project network.

Numbering the Events (Fulkerson's Rule)

1. The initial event which has all outgoing arrows with no incoming arrow is numbered "1".
2. Delete all the arrows coming out from node "1". This will convert some more nodes into initial events. Number these events as 2, 3, 4, ….
3. Delete all the arrows going out from these numbered events to create more initial events. Assign the next numbers to these events.
4. Continue until the final or terminal node, which has all arrows coming in with no arrow going out is numbered.

Determination of time to complete each activity:
The CPM system of networks omits the probabilistic consideration and is based on a Single Time Estimate of the average time required to execute the activity.

In PERT analysis, there is always a great deal of uncertainty associated with the activity durations of any project. Therefore, t_e estimated time is better described by a probability distribution than by a single estimate. Three time estimates (*from beta probability distribution*) are made as follows:

1) The Optimistic Time Estimate (t_o): Shortest possible time in which an activity can be completed in ideal conditions. No provisions are made for delays or setbacks while estimating this time.
2) The Most Likely Time (t_m): It assumes that things go in normal way with few setbacks.
3) The Pessimistic Time (t_p): The max. possible time if everything go wrong & abnormal situations prevailed. However, major catastrophes such as earthquakes, labour troubles, etc. are not taken into account.

The expected time (mean time) for each activity can be approximated using the weighted average i.e.

$$\textbf{Expected Time } (t_e) = \frac{t_o + 4t_m + t_p}{6}$$

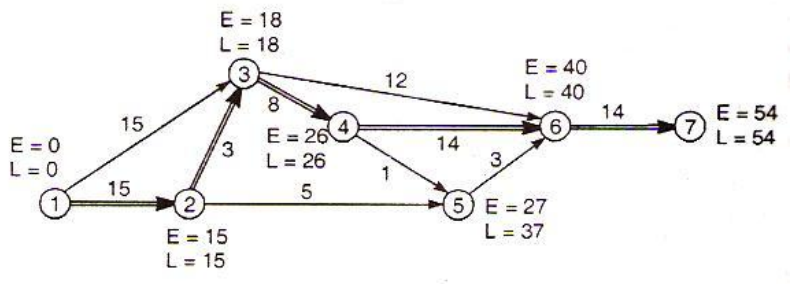
Forward Pass Computation: It is the process of tracing the network from START to END. It gives the earliest start & finish times for each activity.

Earliest event time (E_j): The time that event *j* will occur if the preceding activities are started as early as possible. E_j is the **max**. of the **sums** $E_i + t_{ij}$ involving each immediately precedent event *i* & intervening event *ij.*

Backward Pass Computation: It is the process of tracing the network starting from LAST node & moving backward.

Latest event time (L_j): The latest time that event *i* can occur without delaying completion of project beyond its earliest time. L_i is the **min**. of the **differences** $L_i - t_{ij}$ involving each immediately precedent event *j* & intervening event *ij.*
E.g. of Earliest event time & Latest event time:

Activity:	1-2	1-3	2-3	2-5	3-4	3-6	4-5	4-6	5-6	6-7
Duration (weeks):	15	15	3	5	8	12	1	14	3	14



- In accordance with Zero Slack Convention, if no schedule date for completion of the project is specified, then we take L=E for the terminal event of the project.
- It is a convention to keep the earliest allowance time of the START event as zero.
- Flexibility of non critical in case of event is known as slack & in case of activity is term as float.(Though some writers have used these terms interchangeably).

The critical path can be identified by determining the following four parameters for each *activity*:

- **EST** - earliest start time: the earliest time at which the activity can start given that all its precedent activities must be completed first = E_i
- **EFT** - earliest finish time, equal to the earliest start time for the activity plus the time required to complete the activity = **EST(i-j) + t_{ij}**
- **LFT** - latest finish time: the latest time at which the activity can be completed without delaying (beyond its targeted completion time) the project = L_j
- **LST** - latest start time, equal to the latest finish time minus the time required to complete the activity **= LFT(i-j) - t_{ij}**

CRITICAL PATH: The **critical path** is the path through the project network in which none of the activities have float (*total float is zero*) i.e. A critical path satisfies following 3 conditions:

- EST =LST
- EFT=LFT
- $E_j – E_i = L_j – L_i = t_{ij}$

The duration of project is fixed by the time taken to complete the path through the network with the greatest total duration. This path is known as critical path & activities on it are known as critical activities. A delay in the critical path delays the project. Similarly, to accelerate the project it is necessary to reduce the total time required for the activities in the critical path.

Dummy Activity may or may not lie on critical path.

Question 10 (Critical Path): Tasks A, B, C,….., H, I constitute a project. The precedence relationships are
A<D; A<E; B<F; D<F; C<G; C<H; F<I; G<I

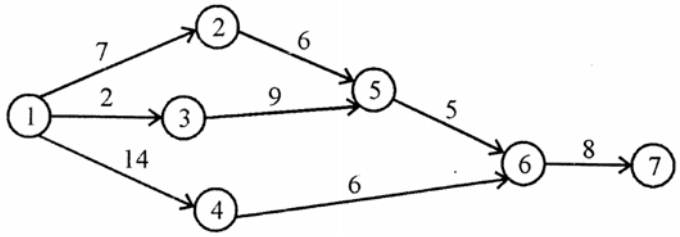
Draw a network to represent the project and find the minimum time of completion of the project when time, in days, of each task is as follows:

Task:	A	B	C	D	E	F	G	H	I
Time:	8	10	8	10	16	17	18	14	9

Also identify the critical path.
[Ans.: Critical path is 1-2-4-5-6 with 44 days**]**

Question 11: At the end of activity 6-7, a product is to be launched and the date has been announced for the inaugural function, based on the normal duration of activities as given in the network below. Activities have

been subcontracted by the project manager to contractors A, B, C, D, E, F, G and H as indicated in the table below. Each subcontractor offers a discount on his contract price for each day given to him in addition to the normal days indicated in the network. What will be the maximum discount that the project manager may earn for the company without delaying the launch of the product?



Activity	Contractor	Discount (₹)/ Day
1 – 2	A	300
1 – 3	B	200
1 – 4	C	1,200
2 – 5	D	500
3 – 5	E	400
4 – 6	F	1,000
5 – 6	G	600
6 – 7	H	500

[Ans.: Maximum discount, without delaying the launch of the product, will be of ₹2600 on extending activity 2-5 by 2 days & 3-5 by 4 days**]** (7 Marks) Nov./10-O.C.

Question 12: A project consists of seven activities for which relevant data are given below:
(i) Draw the network (5 Marks)
(ii) Name and highlight the critical path. (1 Mark) Nov./06& Nov/96[Adapted]

Activity	Preceding Activity	Activity Duration (Days)
A	-	4
B	-	7
C	-	6
D	A,B	5
E	A,B	7
F	C,D,E	6
G	C,D,E	5

[Ans.: B, E, F = 20 days**]**

Question 13 (Forward and Backward Pass): A project schedule has the following characteristics:

Activity	Time (weeks)	Activity	Times (week)
1-2	4	5-6	4
1-3	1	5-7	8
2-4	1	6-8	1
3-4	1	7-8	2
3-5	6	8-9	1
4-9	5	8-10	8
		9-10	7

(i) Construct the PERT network
(ii) Compute E and L for each event;
(iii) *Float for each activity*; and
(iii) Find critical path and its duration. (10 Marks) May/2000

[**Note**: Float is to be calculated only after going through below text]

[Ans.: Critical path is 1-3-5-7-8-10 with 25 weeks**]**

The ***total*** *float time* for an *activity* is the time between its earliest and latest start time, or between its earliest and latest finish time. It is the amount of time that an activity can be delayed past its earliest start or earliest finish without delaying the project. = **LST-EST or LFT-EFT** = LFT-EST-t_{ij} = LFT- (EST+t_{ij})

The slack time or ***slack*** of an *event* in a network is the difference the latest event time & earliest event time i.e. $\mathbf{L_i\text{-}E_i}$

The ***free*** *float time* of an *activity* is equal to the amount by which its duration can be increased without affecting either the project time or the time available for the subsequent activities. It indicates the value by which an activity can be delayed beyond the earliest starting point without affecting the earliest start, & therefore, the total float of the activities following it. **= Total Float$_{ij}$ – (Slack of event *j*)**

The ***independent*** *float time* of an *activity* is the amount by which the duration of an activity could be extended without affecting the total project time, the time available for subsequent activities or the time available for the preceding activities. **= [Free Float$_{ij}$ – (Slack of event i)] or ZERO, whichever is higher**. Also EST of following activity – LFT of preceding activity – Duration of current activity or Zero, whichever is higher.

The ***interfering float time*** is the part of total float which causes a reduction in the float of successor activities. It is that portion of the activity float which cannot be consumed without affecting adversely the float of the subsequent activity or activities. **= LFT – (EST of following activity) or ZERO, whichever is higher.**

> While calculating floats, for just for our simplifying computations, we can write values of Slack of event j in column wherein we are supposed to write interfering float.

Negative Float in a PERT Network: In a CPM network, it is assumed that L=E at the final event but in PERT network, there may be a scheduled date at which a project is expected to complete. This value of the scheduled time, T_s, is often taken as the time for the final event.
When L < E, a negative float results.
In such cases, critical activity is not specified by the condition of zero float along the path. The critical path is the path of least float for such kind of PERT network in which backward pass is based on the scheduled date.

Subcritical Activity: Activity having next higher float than the critical activity.

Supercritical Activity: These Activities have negative float. It results when activity duration is more than time available. It indicates abnormal situation requiring as to how to compress the activity.

Subcritical path: The path with the next least floats than critical path is subcritical path.

Float Calculation

Event Oriented
Here Activities are shown as i-j, where i is tail event & j is head event e.g. 1-2, 2-3, etc. here activity is between 1-2 though starting/tail event(i) is 1 & head/ending event(j) is 2.

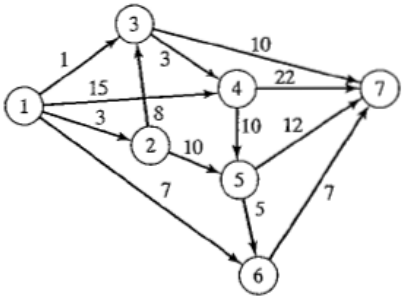
Activity Oriented
Here Activities are generally shown via alphabets e.g. A, B, C, etc.

Question 14 (Floats): The utility data for a network are given below. Determine the total, free, independent and interfering floats and identify the critical path.

Activity:	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7
Duration:	2	8	10	6	3	3	7	5	2	8

[Ans.: Critical Path is 0-1-3-6-7 with 27**]**

Question 15: For the network given below, compute E and L for each event & determine the total, free, independent and interfering floats and identify the critical path.



Question 16: The following table gives the activities in a construction project and the time duration of each activity:

Activity	Preceding activity	Normal Time (Days)
A	-	16
B	-	20
C	A	8
D	A	10
E	B, C	6
F	D, E	12

Required:
(i) Draw the activity network of the project.
(ii) Find critical path.
(iii) Find the total float and free-float for each activity. (6 Marks) Nov/07
[Ans.: (ii) A-C-E-F = 42 days.(iii) Total Float A-0, B-4, C-0, D-4, E-0, F-0; Free Float A-0, B-4, C-0, D-4, E-0, F-0**]**

Question 17: Given is the following information regarding a project:

Activity	A	B	C	D	E	F	G	H	I	J	K	L
Dependence	-	-	-	AB	B	B	FC	B	EH	EH	CDFJ	K
Duration (days)	3	4	2	5	1	3	6	4	4	2	1	5

Draw the Network Diagram and identify the Critical Path and Project Duration. (12 Marks) Nov./94
Find the three types of float (viz. Total, Free and Independent) for each activity.
[Ans.: B-H-J-K-L = 16 days**]**

<u>Probability Estimate</u>: It is used to calculate the probability of completing the time within given duration (*Using Normal Distribution*):

$$Z = \frac{T_1 - T_{cp}}{\sigma_t}$$

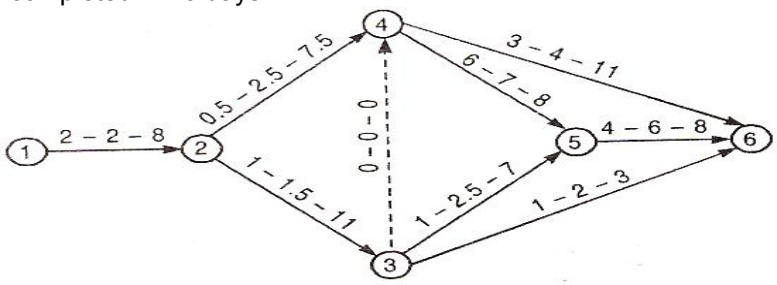
Where, Z = Standard Normal Variate
T_1 = Duration in which we wish to complete the project
T_{cp} = Duration on critical path
σ_t = Standard Deviation of the earliest finish of network = Square root of sum of variance of all activity durations of critical path, where

$$\text{Variance Distribution, } (\sigma^2_t) = \left[\frac{t_p - t_o}{6}\right]^2$$

Question 18: If the critical path of a project is 20 months alongwith a standard deviation of 4 months, what is the probability that the project will be completed within: (a) 20 months (b) 18 months (c) 24 months?
[Ans.: 0.50, 0.31, 0.84**]**

Question 19: PERT calculation yield a project length of 60 weeks with variance of 9. Within how many weeks would you expect the project to be completed with probability of 0.99? (That is the project length that you would expect to be exceeded only by 1% of time if the project were repeated many time in an identical manner).
[Ans.: 67 weeks**]**

Question 20 [Calculation of variance in 2 critical paths]: Consider the network shown below. The three time estimates for the activities are given along the arrows. Determine the critical path. What is the probability that the project will be completed in 20 days?



[Ans.: 0.6844**]**

- In case there are two critical paths, variance of separate activities of both of them shall be added for calculating σ_t, but for calculating Z, we will take higher of two σ_t taken above (As in above question).
- In case of event variance, if there are two longest paths, higher of the two would picked up.

Question 21: Consider the schedule of activities and related information as given below, for the construction of a plant:

Activity	Expected Time (Months)	Variance	Expected Cost (Millions of ₹)
1-2	4	1	5
2-3	2	1	3
3-6	3	1	4
2-4	6	2	9
1-5	2	1	2
5-6	5	1	12
4-6	9	5	20
5-7	7	8	7
7-8	10	16	14
6-8	1	1	4

Assuming that the cost and time required for one activity is independent of the time an cost of any other activity are expected to follow normal distribution.
Draw a network based on the above data and calculate:

(a) Critical path
(b) Expected cost of construction of the plant.
(c) Expected time required to build the plant.
(d) The standard deviation of the expected time. (10 Marks) May/01

[Ans.: (a) 1-2-4-6-8; (b) ₹80 millions; (c) 20 months; (d) 3 months**]**

Question 22: The time estimate (in weeks) for the activities of a PERT network are given below:

Activity	t_o	t_m	t_p
1-2	1	1	7
1-3	1	4	7
1-4	2	2	8
2-5	1	1	1
3-5	2	5	14
4-6	2	5	8
5-6	3	6	15

(a) Draw the project network and identify all the paths through it.
(b) Determine the expected project length.
(c) Calculate the standard deviation and variance of the project length.
(d) What is the probability that the project will be completed.

1. at least 4 weeks earlier than expected time?
2. no more that 4 weeks later than expected time?

(e) If the project due date is 19 weeks, what is the probability of not meeting the due date?
(f) The probability than the project will be completed on schedule if the scheduled completion time is 20 weeks.
(g) What should be the scheduled completion time for the probability of completion to be 90%? (Nov./91)

[Ans.: (c) Variance = 9 & Standard Deviation = 3; (d) (i) 0.0918 (ii) 0.9082 (e) 0.2514 (f) 0.8413 (g) 20.84**]**

Question 23: A company is launching a new product and has made estimates of the time for the various activities associated with the launch as follows:

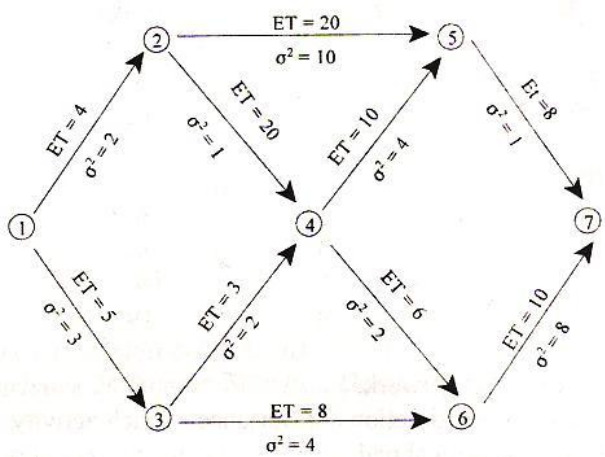
Activity	Predecessor	Times (Days) Optimistic	Most Likely	Pessimistic
A	NONE	1	3	5
B	NONE	3	4	5
C	A,B	1	3	11
D	B	3	3	9
E	A	1	2	3
F	C	2	5	14
G	E,F	2	3	4
H	D,F	2	2	2
I	G,H	10	10	10

Required :
i) Draw the network diagram.
ii) Calculate the expected time and variance of each activity.
iii) Find out the expected length of critical path and its standard deviation.
iv) Find the probability that the launching will be completed in 27 days.
v) Find the duration, which has 95% probability of completion. (8 Marks) Nov./09-O.C.

[Ans.: (iii) Critical Path is B-C-F-G-I with expected project duration of 27 days & standard deviation of 2.646 days; (iv) 50% (v) 31 days approx.**]**
[Note: Answer suggested by ICAI for this question consist myriads of errors.**]**

Question 24: Given the following project network, determine:
1. Earliest expected completion time for each event
2. Latest allowable completion time for each event
3. Slack time for each event
4. Critical Path
5. The probability that project will be completed on schedule, if scheduled completion time is 38

Project Diagram



(8 Marks) Nov./04

[**Hint:** Critical Path is 1-2-4-5-7 and probability = 7.93%]

Question 25: A project consists of seven activities and the time estimates of the activities are furnished as under:

Activity	Optimistic Days	Most likely Days	Pessimistic Days
1-2	4	10	16
1-3	3	6	9
1-4	4	7	16
2-5	5	5	5
3-5	8	11	32
4-6	4	10	16
5-6	2	5	8

Required:

(i) Draw the network diagram.

(ii) Identify the critical path and its duration.

(iii) What is the probability that project will be completed in 5 days earlier than the critical path duration?

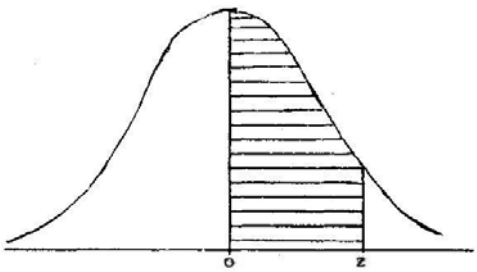
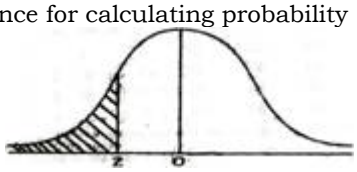
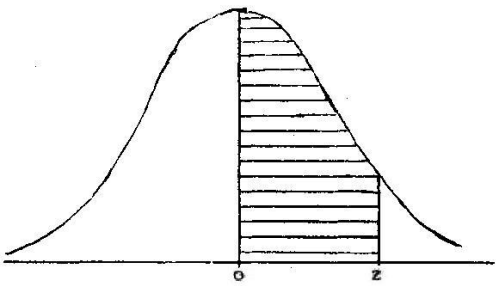
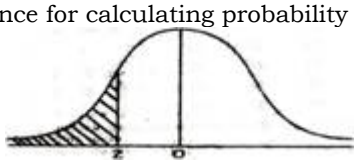
(iv) What project duration will provide 95% confidence level of completion ($Z_{0.95}$ =1.65)?

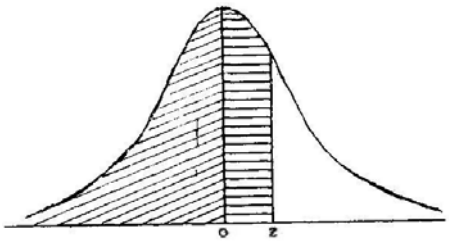
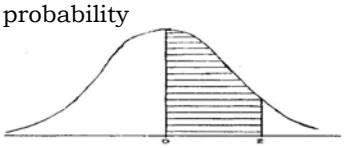
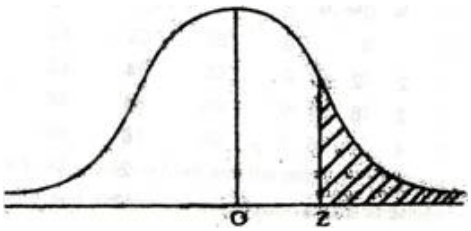
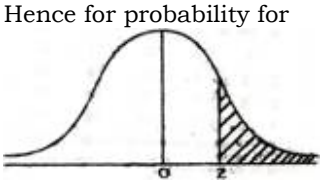
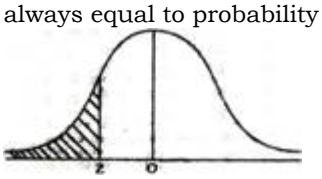
Given (11 Marks) Nov/08-O.C.

Z	1.00	1.09	1.18	1.25	1.33
Probability	0.1587	0.1379	0.1190	0.1056	0.0918

[Ans.: (ii)Critical Path is 1→3→5→6 & its duration is 25 days (iii) Probability = 11.90%, (iv) 32 days (approx)**]**

Remember, while doing questions like above,

whenever we have been provided with a normal table as an extra sheet in examination [generally a green colour sheet], we would calculate probability, treating that probability has been provided for		Hence for calculating probability for  We would check value of z (e.g. 1.18) in the extra sheet table and find probability (i.e. 0.3810), now we will reduce this value from 0.5 to get the probability of the abovesaid region (i.e. 0.5-0.3810 = 0.1190)
But, if in spite of extra sheet of normal table, value of z has been provided as a part of question itself, (as it was in above question)		
If probability is increasing with increase in value of z but all probabilities are less than 0.5, then area has been provided for		Hence for calculating probability for  We would check value of z (e.g. 1.18) in the values given with the question and find probability (i.e. 0.3810), now we will reduce this value from 0.5 to get the probability of the abovesaid region (i.e. 0.5-0.3810 = 0.1190)

If probability is increasing with increase in value of z but all probabilities are greater than 0.5, then area has been provided for		Hence for calculating probability we would first reduce value of probability by 0.5 to make get probability for  Now, we will do the same steps as done in above case.
But if probability is decreasing with increase in value of z (as in above question), then area has been provided for		Hence for probability for  is always equal to probability for  We will simply check value of z (i.e. 1.18) in the value in the question and ***<u>its probability itself will be our answer i.e. 0.1190)</u>***

Question 26: A project consists of the following activities, whose time estimates are given against each as under:

Activity	Estimated duration (weeks) Optimistic	Most likely	Pessimistic
1-2	3	6	15
1-3	2	5	14
1-4	6	12	30
2-5	2	5	8
2-6	5	11	17
3-6	3	6	15
4-7	3	9	27
5-7	1	4	7
6-7	4	19	28

Required :

(i) Draw the project net work.
(ii) Find the expected duration and variance of each activity.
(iii) Determine the critical path and the expected project duration.
(iv) What is the probability that the project will be completed in 38 weeks?
(v) What project duration will have 95% chance of completion. ($Z_{0.95}$ = 1.65)

Given : Z 0.21 0.41 0.82 (8 Marks) May/03
0.0832 0.1591 0.2939

[Ans.: (iii) Critical path 1→2→6→7, Expected project duration is 36 weeks. (iv) 66% (v) 44 weeks**]**

Question 27: An Engineering Project has the following activities, whose time estimates are listed below:

Activity (i-j)	Estimated Duration (in months) Optimistic	Most Likely	Pessimistic
1-2	2	2	14
1-3	2	8	14
1-4	4	4	16
2-5	2	2	2
3-5	4	10	28
4-6	4	10	16
5-6	6	12	30

(a) Draw the project network and find the critical path.
(b) Find the expected duration and variance for each activity. What is the expected project length?
(c) Calculate the variance and standard deviation of the project length.
(d) What is the probability that the project will be completed at least eight months earlier than expected time?
(e) If the project due date is 38 months, what is the probability of not meeting the due date?

Given:					
z	0.50	0.67	1.00	1.33	2.00
P	0.3085	0.2514	0.1587	0.0918	0.0228

[Ans.: (i) Critical path 1-3-5-6; (ii) Expected project length 34 months; (iii) Variance 36 months, Standard Deviation 6; (iv) 9.18%; (v) 25.14%**]** (4 Marks)May/10-NC[(i) & (ii) part],(10 Marks)Nov./01 & (7 Marks)Nov/05

Question 28: The following information is given:

Activity	(1-2)	(2-3)	(2-4)	(3-5)	(4-6)	(5-6)	(5-7)	(6-7)
Pessimistic time (in weeks)	3	9	6	8	8	0	5	8
Most likely time (in weeks)	3	6	4	6	6	0	4	5
Optimistic time (in weeks)	3	3	2	4	4	0	3	2

Draw the Network diagram for the above. Calculate:
1) Variance of each activity.
2) Critical path and expected project length.
3) The probability that the project will be completed in 23 weeks.

Given that:					
Z value:	1.90	1.91	1.92	1.93	1.94
Probability:	0.9713	0.9719	0.9726	0.9732	0.9738

[Ans.: (i) 0, 1, 4/9, 4/9, 4/9, 0, 1/9, 1; (ii) 20 weeks; (iii) 97.26%**]** (10 Marks) May/98

Question 29: A civil engineering firm has to bid for the construction of a dam. The activities and their time estimates are given below:

Activity	Optimistic	Most likely	Pessimistic
1-2	14	17	25
2-3	14	18	21
2-4	13	15	18
2-8	16	19	28
3-4 (dummy)	0	0	0
3-5	15	18	27
4-6	13	17	21
5-7 (dummy)	0	0	0
5-9	14	18	20
6-7 (dummy)	0	0	0
6-8 (dummy)	0	0	0
7-9	16	20	41
8-9	14	16	22

The policy of the firm with respect to submitting bids is to bid the minimum amount that will provide a 95% of probability of at best breaking-even. The fixed costs for the project are eight lakhs and the variable costs are 9000 every day spent working on the project. The duration is in days and the costs are in rupees.
What amount should the firm bid under this policy? (You may perform the calculations on duration etc., up to two decimal places) (May/90)

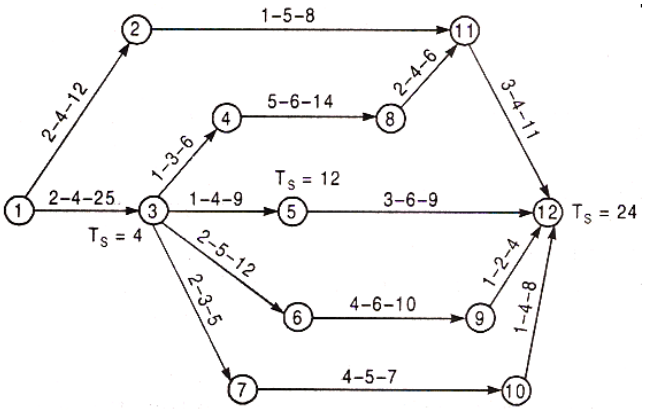
[Ans.: ₹1574000**]**

Question 30 [Negative float-LEAST IMPORTANT]: The optimistic, most likely and pessimistic times of the activities of a project are given below. Activity 40-50 must not start before 22 days, while activity 70-90 must end by 35 days. The scheduled completion time of the project is 46 days. Draw the network and determine the critical path. What is the probability of completing the project in scheduled time?

Activity	t_o-t_m-t_p	Activity	t_o-t_m-t_p
10-20	4-8-12	50-70	3-6-9
20-30	1-4-7	50-80	4-6-8
20-40	8-12-16	60-100	4-6-8
30-50	3-5-7	70-90	4-8-12
40-50	0-0-0	80-90	2-5-8
40-60	3-6-9	90-100	4-10-16

[Ans.: 10-20-40-50-70-90-100 = 46 days; Probability is 50%**]**

Question 31 [Probability at non-critical event- Calculation for this question is very tedious]: A PERT network is shown below. The activity times in days are given along with the arrows. The scheduled times for some important events are given along the nodes. Determine the critical path and probabilities of meeting the scheduled dates for the specified events. Tabulate the results and determine slack for each event.



[Ans.: Critical Path is 1-3-4-8-11-12 with project completion time as 26.51 days, Probability of completing the project in the scheduled completion time of 24 days is 29.54%, Probability that event 3 will occur on scheduled date is 20.44% & Probability of meeting schedule date of event 5 is less than or equal to 54.89% with minimum of 1.8%**]**

Question 32: Distinction between PERT and CPM (5 Marks) Nov/98 & (5 Marks) Nov/07

PERT	CPM
1. PERT is used for non-repetitive jobs like planning the assembly of the space. 2. it is a probabilistic model. 3. It is event-oriented as the results of analysis are expressed in terms of events or distinct points in time indicative of progress. 4. It is applied mainly for planning and scheduling research programmes. 5. PERT incorporates statistical analysis and thereby determines the probabilities concerning the time by which each activity or entire project would be	1. CPM is used for repetitive job like building a house 2. It is a deterministic model. 3. It is activity-oriented as the result or calculations are considered in terms of activities or operations of the project. 4. It is applied mainly for construction and business problems. 5. CPM does not incorporate statistical analysis in determining time estimates, because time is precise and known. 6. It is difficult to use CPM as a control device for the

completed. 6. PERT serves as useful control device as it assists management in controlling a project by calling attention to such delays	simple reason that one must repeat the entire evaluation of the project each time the changes are introduced into the network.

Question 33: A Project Manager has to manage various projects. For each project given below, you are required to advise him whether to use PERT or CPM and briefly state the reason:

(i) Project K is yet to begin. The manager has recently successfully handled similar projects. He is able to break down the project into smaller modules and knows when he may comfortably finish each module.

(ii) Project L has been sanctioned some fixed amount. Though the manager is familiar about what time it will take, he expects pressure towards the end to finish the project slightly earlier, by deploying additional resources of the company.

(iii) Project M is new to the manager. He has never handled such a project. He can break up the project into smaller modules, but even then, he is not sure of their exact times.

(iv) Project N has a limitation on the skilled workforce available. But the manager knows from earlier experience, the slack on each event in the project. He is confident of handling the bottleneck of labour.

(v) Project O is a research project, bound to produce immense benefit to the company in future.

[Ans.: (i) CPM; (ii) CPM; (iii) PERT; (iv) CPM; (v) PERT**]** (5 Marks) May/10-O.C.

Project Crashing

There are usually compelling reasons to complete the project earlier than the originally estimated duration of critical path computed on the normal basis of a new project.

Direct Cost: This is the cost of the materials, equipment and labour required to perform the activity. When the time duration is reduced the project direct cost increases.

Indirect Cost: It consists of two parts: fixed cost and variable cost. The fixed cost is due to general and administrative expenses, insurance, etc. Variable indirect cost consists of supervision, interest on capital, etc.

The total project cost is the sum of the direct & the indirect costs.

Optimum duration is the project duration at which *total* project cost is lowest.

Steps of Project Crashing:

Initialization Steps

Step 1: Prepare Slope table and calculate maximum possible reduction for every activity & cost slope (i.e. crashing cost per unit time) for each activity according to the following formulas:

$$\text{Activity Cost Slope} = \frac{C_c - N_c}{N_t - C_t}$$

Where, C_c = Crash Cost = Direct cost that is anticipated in completing an activity within crash time.

N_c = Normal Cost = This is the lowest possible direct cost required to complete an activity

N_t = Normal Time = Min. time required to complete an activity at normal cost.

C_t = Crash Time = Min. time required to complete an activity.

Maximum Possible reduction (i.e. the number of time periods of crashing availability) = N_t - C_t

Step 2: Enumerate all the paths through the project network, and list them with their normal time durations in the path table. Identify the critical path(s) as those with longest duration, and mark the critical path(s) in Decision table.

Step 3: Identify the normal project duration, the normal project cost and the normal critical path in the first row of the cost table, labeled as iteration 0.

Iteration Steps

Step 4: Select that subset of critical activities which, when compressed in parallel, enable all current critical paths to become shorter, and do so at the least group cost slope, where the group cost slope for a subset of critical activities is the sum of the cost slope for activities in the group.

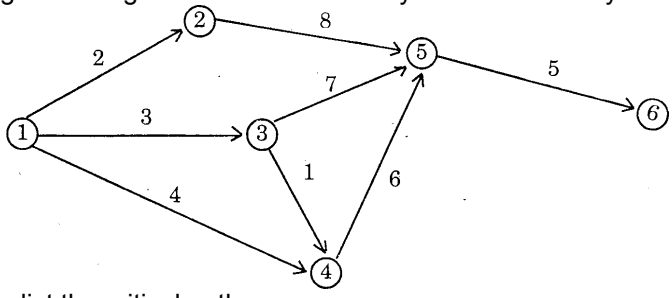
Step 5: Crash the selected critical activities until one or both of the following two conditions occurs: (i) one (or more) of the crashed activities becomes fully crashed (i.e. is reduced to crash time); or (ii) a new path becomes critical.

Step 6: Record the selected activities, number of time periods compressed, the new project duration, the group cost slope for the selected activities, the added cost resulting from the compression, the new total direct cost, and the new critical path (if any) as items in the decision table & cost table for this iteration. Update the slope table and the path table to reflect the reduction in path lengths resulting from the selected crashing.

Step 7: Repeat steps 4 through 6 until all activities on some (any) critical path become fully crashed. At this point the Decision table is complete, as no further time reduction is possible. Plot the time-cost trade-off graph by linear interpolation between the time/cost pairs which occur in each row of the cost table.

> If question has asked to calculate optimum duration & not minimum duration, we will perform crashing only up to the iteration where direct cost slope is equal to indirect cost slope (because at this point total cost will be minimum).

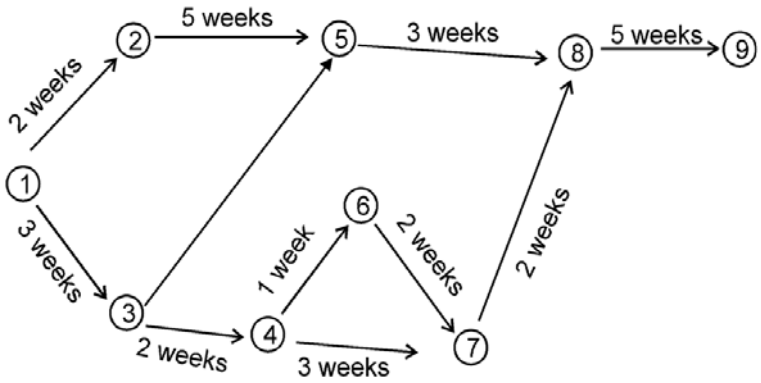
Question 34: The following network gives the duration in days for each activity:



(i) You are required to list the critical paths.
(ii) Given that each activity can be crashed by a maximum of one day, choose to crash any four activities so that the project duration is reduced by 2 days. (6 Marks) Nov./09-N.C.

[Ans.: (i) 1-2-5-6; 1-3-5-6; 1-3-4-5-6; 1-4-5-6 with 15 days; (ii) Either 1-2, 1-3, 1-4, 5-6 or 1-2, 3-5, 4-5, 5-6 or 1-3, 1-4, 2-5, 5-6 or 2-5, 3-5, 4-5, 5-6**]**

Question 35: A network is given below:
(i) Name the paths and give their total duration. (3 Marks)
(ii) Give three different ways of reducing the project above duration by four days. (3 marks) Nov./06



[Ans.: Paths and their duration are 1-2-5-8-9; 1-3-4-7-8-9; 1-3-4-6-7-8-9; 1-3-5-8-9 with 15 weeks**]**
[Hint: Time duration of activity 3-5 was missing in question paper, so best way to assume its duration as exactly 4 weeks, because in that case all paths will become critical paths. You may assume other time durations as well.**]**

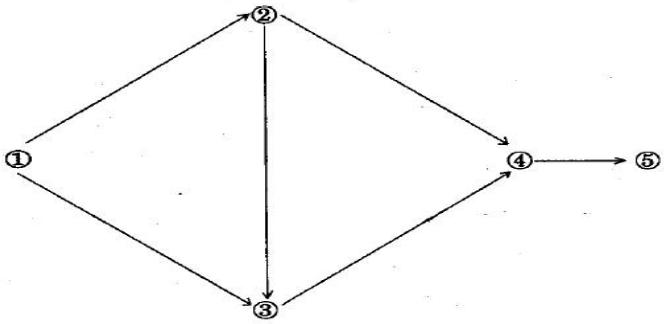
Question 36: A project is composed of seven activities as per the details given below:

Activity	Normal Time (Days)	Crash Time (Days)	Normal Cost (₹)	Crash Cost (₹)
1-2	4	3	1500	2000
1-3	2	2	1000	1000
1-4	5	4	1875	2250
2-3	7	5	1000	1500
2-5	7	6	2000	2500
3-5	2	1	1250	1625
4-5	5	4	1500	2125

Indirect cost per day of the project is ₹500.
Required:
(a) Draw the project network.
(b) Determine the critical path and its duration.
(c) Find the optimum duration and the resultant cost of the project. (8 Marks) May/04

[Ans.: (b) Longest path is 13 days; (c) Optimal project duration of 10 days.**]**

Question 37: The normal time, crash time and crashing cost per day are given for the following network:



Activity	Normal Time (Days)	Crash Time (Days)	Crashing Cost (₹/day)
1-2	18	14	40
1-3	23	22	20
2-3	8	5	60
2-4	10	6	40
3-4	3	2	80
4-5	8	6	50

(i) Crash the project duration in steps and arrive at the minimum duration. What will be the critical path and the cost of crashing?
(ii) If there is an indirect cost of ₹70 per day, what will be the optimal project duration and the cost of crashing?

[Ans.: (i) Min. Duration is 30 days & total cost of crashing is ₹360 (ii) Optimal Duration is 31 days & cost of crashing is ₹280**]** (10 Marks) Nov./08-N.C.

Question 38: A small project consists of jobs as given in the table below. Each job is listed with its normal time and a minimum or crash time (in days). The cost (in ₹per day) for each job is also given:

Job (i-j)	Normal Duration (in days)	Minimum (crash) Duration (in days)	Cost of Crashing (₹per day)
1-2	9	6	20
1-3	8	5	25
1-4	15	10	30
2-4	5	3	10

3-4	10	6	15
4-5	2	1	40

(i) What is the normal project length and the minimum project length?
(ii) Determine the minimum crashing cost of schedules ranging from normal length down to, and including, the minimum length schedule. That is, if L = Length of the schedule, find the costs of schedules which are L, L-1, L-2 and so on.
(iii) Overhead costs total ₹60 per day. What is the optimum length schedule in terms of both crashing and overhead cost? List the schedule duration of each job for your solution. (10 Marks) May/02

[Ans.: (i) Critical path is 1→3→4→5, Normal Project length = 20 days, Minimum Project Length = 12 days; (ii) Optimum total costs = ₹1030; (iii) Optimum duration of the project is 15 days.**]**

Question 39: The following table gives data on normal time and cost and crash time and cost for a project.
(a) Draw the network and identify the critical path.
(b) What is the normal project duration and associated cost?
(c) Find out total float for each activity.
(d) Crash the relevant activities systematically and determine the optimum project time and cost.

Activity	Normal		Crash	
	Time (Week)	Cost (₹)	Time (Week)	Cost (₹)
1-2	3	300	2	400
2-3	3	30	3	30
2-4	7	420	5	580
2-5	9	720	7	810
3-5	5	250	4	300
4-5	0	0	0	0
5-6	6	320	4	410
6-7	4	400	3	470
6-8	13	780	10	900
7-8	10	1000	9	1200
		4220		

Indirect costs are ₹50 per week.

[Ans.: (b) Normal Project Duration is 32 weeks with cost of ₹5820; (d) Optimum Project Duration is 29 weeks with cost of ₹5805**]** (12 Marks) June/09 [Old Course-Adapted] & ICWA Final (Dec-1988)

Question 40: A small project is having seven activities. The relevant data about these activities is given below:

Activity	Dependence	Normal duration (Days)	Crash duration (Days)	Normal cost (₹)	Crash Cost (₹)
A	-	7	5	500	900
B	A	4	2	400	600
C	A	5	5	500	500
D	A	6	4	800	1000
E	B,C	7	4	700	1000
F	C,D	5	2	800	1400
G	E,F	6	4	800	1600

(i) Find out the normal duration and the minimum duration.
(ii) What is the percentage increase in cost to complete the project in 21 days? (10 Marks) Nov./97

[Ans.: (i) Normal duration 25 days, minimum duration 18 days; (ii) 15.5%**]**

Miscellaneous Topics (CPM and PERT)

Question 41: Write short notes on resource smoothing and resource leveling.
(5 Marks each) May/99, May/00, May/02, Nov./02, May/05

Resource smoothing is a resource scheduling technique used for smoothing peak resource requirement during different periods of project network. Under this technique, the constraint may be the total project duration. It helps to estimate the resource requirements for various projects. In resource smoothing, time scaled diagram of various activities of project and their floats along with their resource requirements are used. Floats on non critical activities are utilized & these activities are rescheduled or shifted (while the project duration remains unchanged) so that a uniform demand on resources is achieved.

Resource Levelling (a.k.a. resource allocation) is an operation of resource scheduling wherein constraint may be availability of certain resources. Here project time is varied for maximum utilization of resources i.e. project duration is not treated as an invariant, but the demand on certain specified resources should not go beyond a specified level. The maximum demand of a resource should not exceed the available limit at any point of time. Non critical activities are rescheduled by utilizing their floats.

Points which are worth noting for developing the algorithm for resource allocation:

1. Halt when both resources and activities are available.
2. Prior to allocation at a halt, up date the E.S.T., E.F.T. and float of the activities not allocated at earlier halt time and their succeeding activities. The repercussions may have to be traced right up to the last event.
3. (a) priorities are assigned on the basis of floats e.g. 1st priority to activity with least float, 2nd to the activity with the next higher float and so on.
   (b) In case of tie in floats, assign priorities on the basis of man-days if the activities e.g., 1st priority to the activity with highest M × D.
   (c) In case of tie in man-days even, assign 1st priority to the activity with highest M (gang size).
   (d) In case of tie in M's even, assign 1st priority to the activity with lower i, where i is the tail event number of the activity.
4. When an activity requires more than one man, it may so happen during allocation that the activity requires more number of persons than that available at the halt time under consideration. In such cases, the resources are allocated to the job with next priority for which they are sufficient.
5. During the floating out of activities, the float of an activity may go negative which means that the project duration is going to be extended beyond the critical path. Once the float of an activity becomes negative, there from the float criterion for ascertaining priorities is invalidated. The priorities are then fixed on the basis of M x D, gang size and lower i criteria respectively.

Question 42: The Madras Construction Company is bidding on a contract to install a line of microwave towers. It has identified the following activities, along with their expected time, predecessor restrictions, and worker requirements:

Activity	Duration, Weeks	Predecessor	Crew size, workers
A	4	None	4
B	7	None	2
C	3	A	2
D	3	A	4
E	2	B	3
F	2	B	3
G	2	D,E	3
H	3	F,G	4

The contract specifies that the project must be completed in 14 weeks. This company will assign a fixed number of workers to the project for its entire duration, and so it would like to ensure that the minimum number of workers is assigned and that the project will be completed in 14 weeks. Find a schedule which will do this.

[Hint.: The maximum number of workers to be assigned to the project is 6**]** (10 Marks) May/95

Question 43 (Resource Smoothing): A network with the following activity durations and manpower requirement is given. Analyze the project from point of view of resources to bring out the necessary steps involved in the analysis and smoothing of resources.

Activity:	1-2	2-3	2-4	3-5	4-6	4-7	5-8	6-8	7-9	8-10	9-10
Duration (weeks):	2	3	4	2	4	3	6	6	5	4	4
No. of Men required:	4	3	3	5	3	4	3	6	2	2	9

[Ans.: Demand for men will decrease to 15 to 11**]**

Question 44 (Resource Smoothing): Consider a project consisting of 14 activities having the duration and resource requirement shown below. Analyze the project and smoothen the requirement of the resources.

Activity	Duration (Weeks)	Masons (M)	Labourers (L)
1-2	2	1	2
2-3	3	2	2
2-4	4	3	2
2-5	2	1	3
3-10	4	2	2
4-6	2	3	2
4-7	4	3	3
5-9	4	5	3
6-8	2	1	2
7-9	5	1	3
8-9	3	-	4
9-11	2	1	1
10-11	3	1	2
11-12	2	1	2

[Ans.: Demand for masons will decrease to 8 & 10 for Labourers]

Question 45 (Resource Allocation): The following information is available:

Activity		No. of days	No. of men reqd. per day
A	1-2	4	2
B	1-3	2	3
C	1-4	8	5
D	2-6	6	3
E	3-5	4	2
F	5-6	1	3
G	4-6	1	8

(a) Draw the network and find the critical path.
(b) What is the peak requirement of Manpower? On which day(s) will this occur?
(c) If the maximum labour available on any day is only 10, when can the project be completed?

[Ans.: (i) Critical Path is AD = 10 days (ii) Peak requirement is 11 men, required on days 7 and 9 (iii) the project can be completed in 11 days.**]** (9 Marks) May/08

Question 46 (Resource Allocation): For a project consisting of several activities, the durations and required resources for carrying out each of the activities and their availabilities are given below:
(a) Draw the network, identify critical path and compute the total float for each of the activities.
(b) Find the project completion time under the given resource constraints.

Resources required

Activity	Equipment	Operators	Duration (Days)
1-2	X	30	4
1-3	Y	20	3
1-4	Z	20	6
2-4	X	30	4
2-5	Z	20	8
3-4	Y	20	4
3-5	Y	20	4
4-5	X	30	6

Resource availability:
No. of operators = 50, equipment X = 1, equipment Y = 1, equipment Z = 1

[Ans.: Critical Path is 1-2-4-5 with duration of 14 days, Project requires 21 days for completion under given constraints**]** (Nov./85)

Questions on Resource Smoothing and Resource Leveling are rarely being asked in Examination.

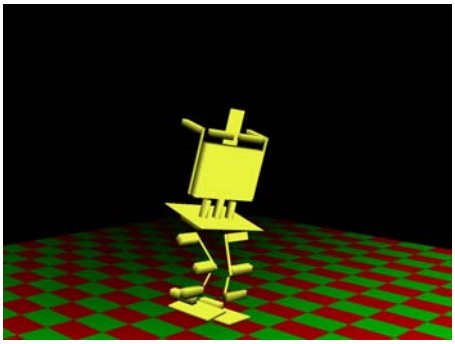
Question 47 (Updating): A company had planned its operations as follows:

Activity:	1-2	2-4	1-3	3-4	1-4	2-5	4-7	3-6	5-7	6-8	7-8
Duration (Days):	7	8	8	6	6	16	19	24	9	7	8

(i) Draw the network and find the critical paths.
(ii) After 15 days of working, the following progress is noted:
(a) Activities 1-2, 1-3 and 1-4 completed as per original schedule.
(b) Activity 2-4 is in progress and will be completed in 4 more days.
(c) Activity 3-6 is in progress and will need 17 more days to complete.
(d) The staff at activity 3-6 is specialized. They are directed to complete 3-6 and undertake an activity 6-7, which will require 7days. This rearrangement arose due to a modification in a specialization.
(e) Activity 6-8 will be completed in 4 days instead of the originally planned 7 days.
(f) There is no change in the other activities.
Update the network diagram after 15 days of start of work based on the assumption given above. Indicate the revised critical paths alongwith their duration. (11 Marks) May/07

[Ans.: Critical Path 1→2→4→7→8 = 42 days; (ii) Critical path 1→3→6→7→8 = 47 days**]**

14



Simulation

According to Shannon "Simulation is the process of designing a model of the real system by conducting experiments with this model for the purpose of designing a model of the real system." *It is imitation of reality*. Maruti car training center with various crossings and signals is a simulated model of city traffic system.

The imitation of reality which may be in the physical form or in the form of mathematical equations may be called simulation. The aircraft simulator to train pilots on new models of aircraft represents the use of physical model as a means of experimentation. The mathematical models of real-life situations in investment analysis, scheduling or inventory control can be experimented; these are known as symbolic simulation models. These models can be either deterministic or probabilistic. The deterministic models can provide answers to 'what if' type of questions in business problems. The probabilistic simulation models deal with random phenomenon and the method of simulation applied is known as Monte Carlo Simulation.

Application of simulation in practical situations

Simulation is quite versatile & commonly applied technique for solving decision problems. It has been applied successfully to a wide range of problems, as given below:

1. Locating of ambulances.
2. Study of projects involving risky investments.
3. Effect of environment on health & group behaviour.
4. Design of computer systems of future needs.
5. Design of weapon systems, war strategies & tactics.
6. New Shop floor management before it is implemented.
7. Design of queuing systems.
8. Design of optimal replenishment policy.
9. Scheduling of production process.
10. Aircraft management & routing.
11. Scheduling of bank tellers & location of bank branches.
12. Routing & dispatching when roads are not secured.
13. Deployment of fire stations & many other problems.

Advantages

1. Behavioral Analysis - Simulation can be used to investigate the behavior of problems which are too complex to be modeled mathematically.
2. Fairly Simple - The simulation technique is easier to use than mathematical models and is considered superior to mathematical models.
3. Flexible - Simulation models are comparatively flexible and can be modified to accommodate the changing environment of the real situation.
4. Time saving – It can compress the performance of a system over several years involving large calculations into few minutes e.g. the effects of ordering, advertising or other policies over many months or years can be obtained by computer simulation in short time.
5. Imparting training - It has advantageously been used for training the operating and managerial staff in operation of complex plans. It is always advantageous to train people on simulation models before putting into their hands the real system.
6. Foresee Hindrances - Through simulation, management can foresee the difficulties and bottlenecks which may come up due to the introduction of new machines, equipment or process.

Limitations (4 Marks) Nov/02 & (3 Marks) May/04

1. Not an Optimizing technique - Simulation does not produce optimum results. It simply allows us to select the best of the alternative systems examined.

2. Costlier on occasions – It is, by no means, a cheap method of analysis. In a number of situations, such as corporate planning, simulation is comparatively costlier and time consuming.
3. Quantification – In number of situations it is not possible to quantify all the variables that affect the behavior of the system.
4. Time Consuming – On occasions simulation is very much time consuming process.
5. Useful in long term only – Reliable results are possible only if the simulation is continued for a long period.

Monte Carlo Simulation (a.k.a. Computer Simulation)
The Monte Carlo method of simulation was developed by two mathematicians John Von Neuman and Stainslaw Ulam. The technique employs random numbers and is used to solve problems that involve probability and wherein physical experimentation is impracticable and formulation of mathematical formula is impossible.

The *steps* involved carrying out Monte Carlo simulation are:
1. Setting up a probability distribution for variables to be analysed.
2. Building a cumulative probability distribution for each random variables.
3. Generating random numbers and then assigning an appropriate set of random numbers to represent value or range (interval) of values for each random variable. If the cumulative probabilities are in two digits the range of random numbers to be assigned are 00 to 99; and if in three digits, the range is from 000 to 999, and so on.
4. Conducting the simulation experiment using random sampling.
5. Repeating the Step 4 until the required number of simulation runs has been generated.
6. Designing and implementing a course of action and maintaining control.

Question 1: State major reason for using a simulation technique to solve a problem and also describe basic steps in a general simulation process. (4 Mark) Nov/06, (4 Marks) Nov/03 & (5 Marks) June/09-O.C.

Question 2: Write short note on Monte Carlo Simulation.
(1 Mark) Nov/06, (5 Marks) Nov/01 & (1 Mark) Nov/03

Question 3: How would you use Monte Carlo Simulation method in inventory control?
(3 Marks) Nov/03 & (4 Marks) May/08

Question 4: What are the steps involved in carrying out Monte Carlo Simulation model?
(4 Marks) Nov./10-N.C.

Question 5: How can simulation be applied in practical situations? (4 Marks) Nov./10-O.C.

Question 6 (Stock simulation): Bright Bakery keeps stocks of a popular brand of cake. Previous experience indicates the daily demand as given here.

Daily Demand	0	10	20	30	40	50
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Consider the following sequence of random numbers;
48 78 19 51 56 77 15 14 68 09
(i) Using the sequence simulate the demand for next 10 days.
(ii) Find out the stock simulation if owner of the Bakery decides to make 30 cakes every day. Also estimate the daily Average demand for the cakes. (10 Marks) Nov./98 & (10 Marks) Nov./99

[Ans.: Average Daily Demand = 22 cakes.**]**

Question 7: A company manufactures around 200 mopeds. Depending upon the availability of raw materials and other conditions, the daily production has been varying from 196 mopeds to 204 mopeds, whose probability distribution is as given below:

Production/day	196	197	198	199	200	201	202	203	204
Probability	0.05	0.09	0.12	0.14	0.20	0.15	0.11	0.08	0.06

The finished mopeds are transported in a specially designed three storied lorry that can accommodate only 200 mopeds. Use the following 15 random numbers 82, 89,78,24, 53, 61, 18, 45, 04, 23, 50, 77, 27, 54, 10 to simulate the process to find out :

(i) What will be the average number of mopeds waiting in the factory?
(ii) What will be the average number of empty space on the lorry?

[Ans.: (i) 2.8 (ii) 0.27**]** (8 Marks) ICWA Dec/06, (10 Marks) May/98 [Adapted] & (10 Marks) May/99

Question 8 (Queuing Problem): Dr. STRONG is a dentist who schedules all her patients for 30 minutes appointments. Some of the patients take more or less than 30 minutes depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and the time needed to complete the work :

Category	Time required (mts)	Probability of category
Filling	45	0.40
Crown	60	0.15
Cleaning	15	0.15
Extraction	45	0.10
Checkup	15	0.20

Simulate the dentist's clinic for four hours and determine the average waiting time for the patients as well as the idleness of the doctor. Assume that all the patients show up at the clinic at exactly their scheduled arrival time starting at 8.00 A.M. Use the following random numbers handling the above problem:

40 82 11 34 25 66 17 79 (12 Marks) Nov/90

[Ans.: Avg waiting time = 35.625 minutes; Dentist's idle time = nil**]**

Question 9 (Service Sector): A single counter ticket booking centre employs one booking clerk. A passenger on arrival immediately goes on the booking counter for being served if the counter is free. If, on the other hand, the counter is engaged, the passenger will have to wait. The passengers are served on first come first served basis. The time of arrival and the time of service varies from one minute to six minutes. The distribution of arrival and service time is as under:

Arrival/Service Time (Minutes)	Arrival (Probability)	Service (Probability)
1	0.05	0.10
2	0.20	0.20
3	0.35	0.40
4	0.25	0.20
5	0.10	0.10
6	0.05	-

Required:

(i) Simulate the arrival and service of 10 passengers starting from 9 A.M. by using the following random numbers in pairs respectively for arrival and service. Random numbers 60 09 16 12 08 18 36 65 38 25 07 11 08 79 59 61 53 77 03 10.
(ii) Determine the total duration of
(1) Idle time of booking clerk and
(2) Waiting time of passengers

[Ans.: (1) 6 Mins. (2) 6 Mins.**]**
(6 Marks) May/10-O.C.[Adapted], (8 Marks) Nov/08-O.C. & (7 Marks) May/03 [Adapted]

Question 10 (Service Sector): At a small store of readymade garments, there is one clerk at the counter who is to check bills, receive payments and place the packed garments into fancy bags. The arrival of customer at the store is random and service time varies from one minute to six minutes, the frequency distribution for which is given below:

Time between arrivals	Frequency	Service Time	Frequency
1	5	1	1
2	20	2	2
3	35	3	4
4	25	4	2
5	10	5	1
6	5	6	0

The store starts work at 11 a.m. and closes at 12 noon for lunch and the customers are served on the "first came first served basis".
Using Monte Carlo simulation technique, find average length of waiting line, average waiting time, average service time and total time spent by a customer in system.
You are given the following set of random numbers, first twenty for arrivals and last twenty for service :

64	04	02	70	03	60	16	18	36	38
07	08	59	53	01	62	36	27	97	86
30	75	38	24	57	09	12	18	65	25
11	79	61	77	10	16	55	52	59	63

[Ans.: Average length of waiting line is 1.3 customers, Average waiting time is 2.8 mins, Average service time is 2.7 mins; Time a customer spends in system = 2.8 + 2.7 = 5.5 minutes.**]** (9 Marks) Nov./09-O.C.

[Note: *Few terminologies & their meaning in Queuing System:*

(i) Queue length – The average number of customers in the queue waiting to get service. This excludes the customer(s) being served.
(ii) System length – The average number of customers in the system including those waiting as well those being served.
(iii) Waiting time in the queue – The average time for which a customer has to wait in the queue to get service.
(iv) ***Total*** time in the system – the ***average*** time spent by a customer in the system from the moment he arrives till he leaves the system. It is taken to be waiting time + service time.**]**

Question 11: Ramu and Raju are workers on a two-station assemble line. The distribution of activity times at their stations is as follows:-

Time in Sec.	Time frequency for Ramu	Time frequency for Raju
10	4	4
20	6	5
30	10	6
40	20	7
50	40	10
60	11	8
70	5	6
80	4	4

(a) Simulate operation of the line for eight times. Use the random numbers given below:

Operation 1		Operation 2	
14	61	36	97
01	82	76	41
95	00	55	56
44	03	25	34

(b) Assuming Raju must wait until Ramu completes the first item before starting work, will he have to wait to process any of the other eight items? Explain your answer, based upon your simulation.

[Ans.: (b) No**]**

Question 12: The occurrence of rain in a city on a day is dependent upon whether or not it rained on the previous day. If it rained on the previous day, the rain distribution is given by:

Event	Probability
No rain	0.50
1 cm. Rain	0.25
2 cm. Rain	0.15
3 cm. Rain	0.05
4 cm. Rain	0.03
5 cm. Rain	0.02

If it did not rain the previous day, the rain distribution is given by:

Event	Probability
No rain	0.75
1 cm. Rain	0.15
2 cm. Rain	0.06
3 cm. Rain	0.04

Simulate the city's weather for 10 days and determine by simulation the total days without rain as well as the total rainfall during the period. Use the following random numbers:

67 63 39 55 29 78 70 06 78 76

for simulation. Assume that for the first day of the simulation it had not rained the day before.

[Ans.: It didn't rain on 6 out of 10 days. Total rainfall during the period is 5cm.**]** (10 Marks) Nov./93

Question 13: The output of a production line is checked by an inspector for one or more of three different types of defects, called defects A, B and C. If defect A occurs, the item is scrapped. If defect B or C occurs, the item must be reworked. The time required to rework a B defect is 15 minutes and the time required to rework a C defect is 30 minutes. The probabilities of an A, B and C defects are 0.15, 0.20 and 0.10 respectively. For ten items coming off the assembly line, determine the number of items without any defect, the number scrapped and the total minutes of rework time. Use the following random numbers:

RN for defect A	48	55	91	40	93	01	83	63	47	52
RN for defect B	47	36	57	04	79	55	10	13	57	09
RN for defect C	82	95	18	96	20	84	56	11	52	03

[Ans.: No defect in 5 items, one item is scrapped & Rectification time required on reworking is 90 minutes**]**

(10 Marks) May/94 & (7 marks) May/04 [Adapted]

Question 14 (Purchase Quantity Decision): A book-store wishes to carry Systems Analysis and Design in stock. Demand is probabilistic and replenishment of stock takes 2 days (i.e., if an order is placed in March 1, it will be delivered at the end of the day on March 3). The probabilities of demand are given below:

Demand (daily) :	0	1	2	3	4
Probability :	0.05	0.10	0.30	0.45	0.10

Each time an order is placed, the store incurs an ordering cost of ₹10 per order. The store also incurs a carrying cost of ₹0.50 per book per day. The inventory carrying cost is calculated on the basis of stock at the end of each day. The manger of the book-store wishes to compare two options for his inventory decision:

A. Order 5 books, when the inventory at the beginning of the day plus orders outstanding is less than 8 books.

B. Order 8 books, when the inventory at the beginning of the day plus orders outstanding is less than 8 books.

Currently (beginning of the 1st day) the store has stock of 8 books plus 6 books ordered 2 days ago and expected to arrive next day. Using Monte-Carlo simulation for 10 cycles, recommend which option the manager should choose?

The two digits random numbers are given below:

89, 34, 78, 61, 63, 81, 39, 16, 13, 73 (10 Marks) May/00, (7 Marks) Nov./04 [Adapted]

[Ans.: Manager should go for Option B, as its Total Cost ₹42.5 is less than Option A's ₹59.5**]**

[Hint: Assume that the demand occurring during the day can be met out of stock received at the end of the day**]**

Question 15: A company uses a high grade book raw material. The consumption pattern is probabilistic as given below and it takes two months to replenish stocks:

Consumption per month (tons) :	1	2	3	4
Probability :	0.15	0.30	0.45	0.10

The cost of placing an order is ₹1000 and the cost of carrying stocks is ₹50 per month per ton. The average carrying costs are calculated on the stocks held at the end of each month.
The company has two options for purchase of raw material as under:
Option I. Order 5 tons, when the closing inventory of the month plus orders outstanding is less than 8 tons.
Option II. Order 8 tons, when the closing inventory of the month plus orders outstanding is less than 8 tons.
Currently on 1st April 2002, the company has a stock of 8 tons of raw materials plus 6 tons ordered two months ago. The order quantity is expected to be received next month.
Using the random numbers given below, simulate 12 months consumption till 31-3-2003 and advise the company as to which purchase option should be accepted such that the inventory costs are minimum.
Random numbers are:
88, 41, 67, 63, 48, 74, 27, 16, 11, 64, 49, 21

(ICWA Dec. 98-Adapted)&(16 Marks) ICWA Dec/07-Adapted

[Ans.: Co. should go for Option II, as its Total Cost ₹5350 is less than Option I's ₹7200**]**
[Hint: Since raw material is ordered in 3rd month after calculation of closing stock, it will be received on 5th months but after calculation of closing stock. Since closing stock of 5th month & opening stock of 6th month will always be same, hence we will treat that order has been received in 6th month. Similar assumption will apply for other reorder points as well.**]**

Question 16: A Publishing house has brought out a new monthly magazine, which sells at ₹37.5 per copy. The cost of producing it is ₹30 per copy. A Newsstand estimates the sales pattern of the magazine as follows:

Demand Copies	Probability
0 < 300	0.18
300 < 600	0.32
600 < 900	0.25
900 < 1200	0.15
1200 < 1500	0.06
1500 < 1800	0.04

The newsstand has contracted for 750 copies of the magazine per month from the publisher.
The unsold copies are returnable to the publisher who will take them back at cost less ₹4 per copy for handling charges.
The newsstand manager wants to simulate of the demand and profitability. The following random number may be used for simulation:

27, 15, 56, 17, 98, 71, 51, 32, 62, 83, 96, 69.

You are required to-
(i) Allocate random numbers to the demand pattern forecast by the newsstand.
(ii) Simulate twelve months sales and calculate the monthly and annual profit/loss.
(iii) Calculate the loss on lost sales. (8 Marks) Nov./05 & (June/97)ICWA-Adapted

[Ans.: Loss on lost sale = ₹15750**]**
[Hint: In statistical terminology, 0<300 is used for "exclusive method" of class interval & it means 0-300. To calculate demand, we shall find class mid-points/marks i.e. average of upper Limit and lower limit**]**

Question 17 (Perishable Goods): A retailer deals in a perishable commodity. The daily demand and supply are variables. The data for the past 500 days, show the following demand and supply:

Supply		Demand	
Availability (kg.)	**No. of days**	**Demand (kg.)**	**No. of days**
10	40	10	50
20	50	20	110
30	190	30	200

40	150	40	100
50	70	50	40

The retailer buys the commodity at ₹20 per kg. and sells it at ₹30 per kg. Any commodity remains at the end of the day, has no saleable value. Moreover, the loss (unearned profit) on any unsatisfied demand is ₹8 per kg. Given the following pair of random numbers, simulate 6 days sales, demand and profit.

(31,18); (63,84);(15,79);(07,32);(43,75);(81,27).

The first random number in the pair is for supply and the second random number is for demand viz. in the first pair (31,18), use 21 to simulate supply and 18 to simulate demand. (10 Marks) Nov/00

[Ans.: ₹400**]**

Question 18: For a washing powder manufacturing factory, frequency distribution of contribution (=sales price-variable cost) per unit annual demand and requirement of investment were found as follows:

Annual Demand		Contribution per unit		Required investment	
Units	**Probability**	**₹**	**Probability**	**₹**	**Probability**
20000	0.05	3.00	0.10	1750000	0.25
25000	0.10	5.00	0.20	2000000	0.50
30000	0.20	7.00	0.40	2500000	0.25
35000	0.30	9.00	0.20		
40000	0.20	10.00	0.10		
45000	0.10				
50000	0.05				

Consider the random numbers 93,03,51,59,77,61,71,62,99,15 for using Monte-Carlo simulation for 10 runs to estimate the percentage of return on investment (ROI%) defined as

$$\text{ROI \%} = \frac{\text{Cash Inflow}}{\text{Investment}} \times 100$$

Recommend an optimum investment strategy based on ***modal*** value of ROI%.

[Ans.: Modal value of ROI % is 12.25, the optimal investment strategy is to investment ₹2000000**]**

(10 Marks) Nov./95 & (10 Marks) May/01[Adapted]

Question 19: ABC Cooperative Bank receives and disburses different amount of cash in each month. The bank has an opening cash Balance of Rs. 15 crores in the first month. Pattern of receipts and disbursements from past data is as follows:

Monthly Cash receipts		Monthly Cash disbursements	
Rs. In Crores	**Probability**	**Rs. In Crores**	**Probability**
30	0.20	33	0.15
42	0.40	60	0.20
36	0.25	39	0.40
99	0.15	57	0.25

Simulate the cash position over a period of 12 months.

Required:

(i) Calculate probability that the ABC Cooperative Bank will fall short in payments.

(ii) Calculate average monthly shortfall.

(iii) If ABC Bank can get an overdraft facility of Rs. 45 crores from other Nationalized banks. What is the probability that they will fall short in monthly payments?

Use the following sequence (rowwise) of paired random numbers.

1778 4316 7435 3123 7244 4692 5158 6808 9358 5478 9654 0977

[Ans.: (i) 0.83; (ii) Average shortfall is 399 ÷ 12 = Rs. 33.25 crores; Alternatively, average shortfall is 399 ÷ 10 = Rs. 39.90 crores (iii) 0.42**]** (7 Marks) May/10-N.C.

[Hint: Payments may be divided into 2 parts viz. Amount paid & amount due but not paid in current month.**]**

Question 20: The Ever alert Ltd., which has a satisfactory preventive maintenance system in its plant, has installed a new Hot Air Generator based on electricity instead of fuel oil for drying its finished products. The Hot Air Generator requires periodicity shutdown maintenance. If the shut down is scheduled yearly, the cost of maintenance will be as under:

Maintenance cost (₹)	Probability
15,000	0.3
20,000	0.4
25,000	0.3

The costs are expected to be almost linear i.e. if the shut down is scheduled twice a year the maintenance cost will be double.
There is no previous experience regarding the time taken between break downs. Costs associated with break down will vary depending upon the periodicity of maintenance. The probability distribution of break down cost is estimated as under:

Breakdown costs ₹Per annum	Shutdown once a year	Shutdown twice a year
75,000	0.2	0.5
80,000	0.5	0.3
1,00,000	0.3	0.2

Simulate the total costs maintenance and breakdown cost- and recommend whether shutdown overhauling should be restored to once a year or twice a year? (ICWA-Dec/96)

[Ans.: The average annual cost will only be ₹1.06 lakhs as against ₹1.20 lakhs when shutdown is twice a year hence shutdown maintenance/overhauling once a year will be more economical**]**
[Hint: Since random nos. were not provided, assume them at your own. In such answer may vary.**]**

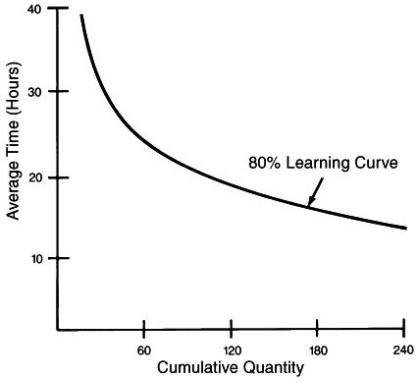
Question 21 (Network simulation-Do after PERT/CPM): A project consists of 7 activities. The time for performance of each of the activity is as follows:-

Activity	Immediate	Time	Probability
A	-	3	0.2
		4	0.6
		5	0.2
B	-	4	1.0
C	A	1	1.0
D	B,C	4	0.8
		5	0.2
E	D	3	0.1
		4	0.3
		5	0.3
		6	0.3
F	D	5	0.2
		7	0.8
G	E,F	2	0.5
		3	0.5

a) Draw a network and identify critical path using expected time.
b) Simulate the project for 5 times using random number and find the critical paths?

68	13	09	20	73	07	72
99	93	18	24	22	07	29
57	33	49	65	92	98	00
57	12	31	96	85	92	91
77	37	34	11	27	10	59

15



Learning Curve Theory

The Theory of learning curve was first introduced by T.P. Wright of Curtiss—Wright, Buffalo, U.S.A. engaged in production of airframes.

CIMA Terminology: The learning curve is "The mathematical expression of the phenomenon that when complex & labour-intensive procedures are repeated, unit labour times tend to decrease at a constant rate. The learning curve models mathematically this reduction in unit production time."

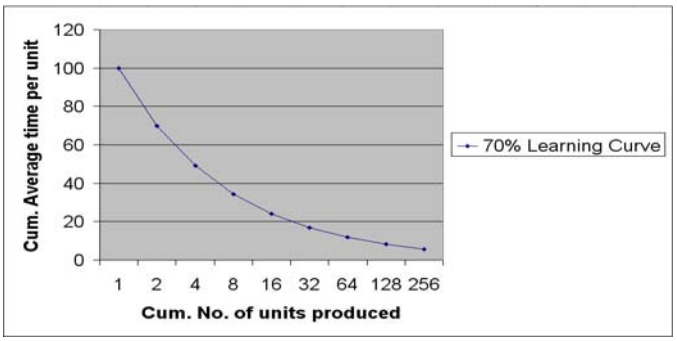
More specifically, the learning curve theory states that the cumulative average time per unit produced is assumed to decrease by a constant percentage every time total output of product doubles. For instance, with an *80% learning curve*, the *cumulative average* time per unit of *output will fall to 80%* of what it was before, every time output is doubled.

E.g. If the first unit of output requires 100 hours & an 70% learning curve applies, the production times would be as follows.:

Cumulative Average-Time Learning Model (Doubling Approach)

Cum. No. of units produced	Cum. Average time per unit (Hrs.)	Total time required (Hrs.)	Incremental time taken (Total Hours)	Incremental units produced	Incremental time taken (Hrs. per unit)
1	100.0	100.0 (100×1)	100	1	100 (100÷1)
2 (1×2)	70.0 (100×70%)	140.0 (70×2)	40	1	40 (40÷1)
4 (2×2)	49.0 (70×70%)	196.0 (49×4)	56	2	28 (56÷2)
8 (4×2)	34.3 (49×70%)	274.4 (34.3×8)	78.4	4	19.6 (78.4÷4)

Graph of Learning Curve:



Cumulative Average-Time Learning Model (Equation Approach)

Cum. No. of units produced (X)	Cum. Average time per unit (Hrs.)[1]	Cumulative Total time required (Hrs.)	Incremental time taken/Individual unit time for X^{th} unit
1	100	100	100
2	70	140	40 (140-100)
3	56.8	170.4	30.4 (170.4–140)
4	49	196	25.6 (196-170.4)
5	43.7	218.5	22.5 (218.5-196)

6	39.8	238.8	20.3 (238.8-218.5)
7	36.7	256.9	18.1 (256.9-238.8)
8	34.3	274.4	17.5 (274.4-256.9)

[1]The mathematical relationship underlying the cumulative average-time learning model is

$$Y_x = aX^b$$

Where, Y = cumulative average time per unit to produce x units.
X = number of units made so far (cumulative no. of units)
a = time for the first unit

b = the learning coefficient or the index of learning = $\frac{\log(\text{learning ratio})}{\log 2}$

(*For e.g.* for 70% learning curve, b = $\frac{\log 0.70}{\log 2} = \frac{-0.1549}{0.3010}$ = -0.5146)

Learning ratio/percentage = $\frac{\text{Average labour cost / time of I}^{st}\text{ 2N units}}{\text{Average labour cost / time of I}^{st}\text{ N units}}$

- Learning Curve Table gives us factor for learning curve, which is used as value of X^b in the above-mentioned formula of learning curve.
- If the initial output was a batch of units rather than just one, the table of factors would be derived in the same way except that we firstly use factor according to batch & hours calculated on individual units.

Uses of Learning curve:

(i) *CVP Analysis:* It helps in analyzing cost-volume profit relationship and is useful for cost estimates and forecasting.
(ii) *Budgeting:* It helps in preparing realistic budgets and profit planning.
(iii) *Pricing:* It helps in pricing, particularly in a tender when it is known that the tender consists of several repetitive jobs.
(iv) *Decisioning:* It helps design engineers in making decision, based upon expected rates of improvement.
(v) *Setting Standards:* It helps in setting standards in learning phase.
(vi) *Human Resource planning:* Its knowledge helps in manpower planning for contract of long duration or for repetitive clerical work.

Limitations of learning curve theory:

1. The learning curve is useful only for new operations where machines do not constitute a major part of the production process. It is not applicable to all productions. E.g. New and experienced workmen.
2. The learning curve assumes that the production will continue without any major interruptions. If for any reason the work in interrupted, the curve may be deflected or assume a new slopes.
3. Changes other than learning may affect the learning curve. For example, improvement in facilities, arrangements, and equipment as well as personnel morale and performance may be factors influencing the curve. On the other hand, negative developments in employee attitudes may also affect the curve and reverse or retard the progress of improvement.
4. The characteristic 80 percent learning curve as originally obtaining in the air force industry in U.S.A. has been usually accepted as the percentage applicable to all industries. Studies show that there cannot be a unique percentage which can be universally applied.
5. Normally, the trade unions will not accept gradual reduction in production time per unit. In this type of situation, management may try to establish a low standard time per unit from the onset. This will lead to adverse efficiency variances, until learning effect has taken place.
6. If there is productivity bonus, incorporation of learning effect may frustrate workers in learning stage, as it may appear as a threat to the size of bonus, which they may earn.

Experience Curve : It is a term applied to the 'corporate embodiment' of the shop floor, managerial & technological learning effects within an organization & it expresses the way in which the average cost per unit of production changes over time due to technological & organizational changes to factory size, product, design, materials used & so on, not just 'learning' by skilled workers.

Important Points in regards to logarithms:
- $Y=x^a$ can also be written as log Y = a log x
- log (x × y) = log x + log y
- log (x ÷ y) = log x – log y
- log 1 = 0, log 10 = 1 & log 100 = 2

Question 1: Explain the concept of 'Learning curve'. How can it be applied for Cost Management?
(2 Marks) Nov/06, (4 Marks) May/06, (4 Marks) May/07 & (4 Marks) Nov/07

Question 2: Explain the concept of Learning curve and discuss its relevance to pricing decisions.
(4 Marks) May/04

Question 3: What are the distinctive features of learning curve theory in manufacturing environment?
(4 Marks) May/03, (5 Marks) Nov/07, (4 Marks) Nov./10-N.C.

Answer: As the production quantity of a given item is doubled, the cost of the item decreases at a fixed rate. This phenomenon is the basic premise on which the theory of learning curve has been formulated. As the quantity produced doubles, the absolute amount of cost increase will be successively smaller but the rate of decrease will remain fixed. It occurs due to the following distinctive features of manufacturing environment:

(a) Better tooling methods are developed and used.
(b) More productive equipments are designed and used to make the product.
(c) Design bugs are detected and corrected.
(d) Engineering changes decrease over time.
(e) Earlier teething problems are overcome.
(f) Rejections and rework tend to diminish over time.

Question 4: The following information is provided by a firm. The factory manager wants to use appropriate average learning rate on activities so that he may forecast costs and prices for certain levels of activity.

i) A set of very experienced people feed data into the computer for processing inventory records in the factory. The manager wishes to apply 80% learning rate on data entry and calculation of inventory.
ii) A new type of machinery is to be installed in the factory. This is patented process and the output may take a year for full fledged production. The factory manager wants to use a learning rate on the workers at the new machine.
iii) An operation uses contract labour. The contractor shifts people among various jobs once in two days. The labour force performs one task in 3 days. The manager wants to apply an average learning rate for these workers.

You are required to advise to the manager with reasons on the applicability of the learning curve theory on the above information. (4 Marks) Nov./09-O.C.

[Ans.: (i), (ii) & (iii) Don't apply Learning curve theory**]**

Question 5: A company which has developed a new machine has observed that the time taken to manufacture the first machine is 600 hours. Calculate the time which the company will take to manufacture the second machine if the actual learning curve rate is (i) 80% and (ii) 90%. Explain which of the two learning rates will show faster learning. (3 Marks) Nov./08-O.C.

[Ans.: (i) 360 hours (ii) 480 hours; 80% is shows faster learning**]**

Question 6: (a)Your company has been approached by a customer to supply four units of a new product made to the customer's individual specification. The company experiences a 90% learning rate. The estimated labour time for the first unit of this product is 150 hours and the company's direct labour cost is ₹5 per hour. Estimate the labour cost for this order.
(b) After receiving the first order, if the customer places a repeat order, what will be the labour cost for the second order.
(c) If the customer had ordered all eight units at the same time, calculate the labour cost per unit for the combined order.

[Ans.: (a) 2430 (b) 1944 (c) 4374**]**

Question 7: A factory has a special offer to produce 4 units of a labour intensive product by using its existing facilities after the regular shift timings. The product can be produced by using only overtime hours which entails normal rate plus 25%, so that usual production is not affected. Two workers are interested in taking up this additional job every evening after their usual shift is over. One is an experienced man who has been working on a similar product. His normal wages are ₹48 per hour. The other worker is a new person who earns ₹42 an hour as normal wages. He can be safely considered to have a learning curve ratio of 90% for this work. The company wants to minimize labour cost for the order and only one person is to be chosen for the job. The experienced man will take 20 hours for the first unit while the new worker will take 30 hours for the first unit. Evaluate who should be chosen for the job. (5 Marks) Nov./10-O.C.
[Ans.: Employing experienced person will lead into savings of ₹303 (5103-4800)**]**

Question 8: The Gifts Company makes mementos for offering chief guests and other dignitaries at functions. A customer wants 4 identical pieces of hand-crafted gifts for 4 dignitaries invited to its function.
For this product, the Gifts Company estimates the following costs for the 1st unit of the product

	₹/unit
Direct variable costs (excluding labour)	2,000
Direct labour (20 hours @ ₹50 per hour)	1,000

90% learning curve ratio is applicable and one labourer works for one customer's order.

i) What is the price per piece to be quoted for this customer if the targeted contribution. is ₹1,500 per unit ?
ii) If 4 different labourers made the 4 products simultaneously to ensure faster delivery to the customer, can the price at (i) above be quoted? Why? (6 Marks) Nov./09-N.C.

[Ans.: (i) ₹4310 per piece (ii) ₹4500 per piece**]**

Question 9: In a particular industry, a 60% learning curve applies. The first item of a new product took 100 hours to make. How long should the 6th item take?

[Ans.: 7.55 hours**]**

Question 10: ABC Co. Ltd. has observed that a 80% learning curve ratio applies to all labour-related costs realating Model X, which has been recently introduced in the market. It is expected that first unit will take a time of 10 minutes. What is the expected time per unit (i) when cumulative production is 20 units, and (ii) when cumulative production is 10 units?

[Ans.: (i) 3.812 minutes & (ii) 4.764 minutes**]**

Question 11: The 1st batch of 50 units of Product A took 80 hours to make. The company now wishes to estimate, what average time per unit will be if the total output for product A is 125 units. An 80% learning curve applies.

[Ans.: 1.2 hours**]**

Question 12: A customer requires separate price quotations for each of the following possible order:

Order	Units
First	100
Second	60
Third	40

It is expected that in first order 4 hours per unit will be consumed. A 80% learning curve applies. Following learning curve table for 80% learning curve is given:

X	1.0	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Y%	100.0	91.7	89.5	87.6	86.1	84.4	83.0	81.5	80.0

Find out average time per unit:

(a) when two orders are expected.
(b) When three orders are expected.

[Ans.: (a) 3.44 (b) 3.2**]**

Question 13: M Ltd. manufactures a special product purely carried out by manual labour. It has a capacity of 20000 units. It estimates the following cost structure:

Direct Material	30 ₹/unit
Direct Labour (1 hour/unit)	20 ₹/unit
Variable overhead	10 ₹/unit

Fixed overheads at maximum capacity are ₹150000.
It is estimated that at the current level of efficiency, each unit requires one hour for the first 5000 units. Subsequently it is possible to achieve 80% learning rate. The market can absorb the first 5000 units at ₹100 per unit. What should be the minimum selling price acceptable for an order of 15000 units for a prospective client? (7 Marks) May/08

[Ans.: Min. Selling Price is ₹50.40**]**

Question 14: PQ Ltd. makes and sells a labour-intensive product. Its labour force has a learning rate of 80%, applicable only to direct labour and not to variable overhead.
The cost per unit of the first product is as follows:

Direct materials	10000	
Direct labours	8000	(@₹4 per hour)
Variable overheads	2000	
Total variable cost	20000	

PQ Ltd. has received an order from X Ltd. for 4 units of the product. Another customer, Y Ltd. is also interested in purchasing 4 units of the product. PQ Ltd. has the capacity to fulfill both the orders. Y Ltd. presently purchases this product in the market for ₹17,200 and is willing to pay this price per unit of PQ's product. But X Ltd. lets PQ choose one of the following options :
(i) A price of ₹16,500 per unit for the 4 units it proposes to takes from PQ.

Or

(ii) Supply X Ltd.'s idle labour force to PQ, for only 4 units of production, with PQ having to pay only Re. 1 per labour hour to X Ltd.'s workers. X Ltd.'s workers will be withdrawn after the first 4 units are produced. In this case, PQ need not use its labour for producing X Ltd.'s requirement. X Ltd. assures PQ that its labour force also has a learning rate of 80%. In this option, X. Ltd. offers to buy the product from PQ at only ₹14,000 per unit.
X and Y shall not know of each other's offer.
If both orders came before any work started, what is the best option that PQ may choose?
Present suitable calculations in favour of your argument. (8 Marks) June/09-N.C.

[Ans.: PQ will earn ₹2832 extra if it chooses Option (i)**]**

Question 15: ABC Ltd. is a company engaged in the provision of small quantities of specialized equipment to industrial customers. The work involved in the manufacture of the output calls for a high level of manual skill, and since no product has a long technological life, it is thought that a 70% learning curve applies to all output made by the company.
A customer has asked for the delivery of eight units of a product which has never before been made. An estimate of the direct cost of the first unit has been derived as follows:

Materials	₹400
Direct labour (250 hours)	1250
	1650

The company adds a contribution margin of 160% on direct cost to cover fixed costs and provide a profit margin.
Required:
(a) State the formula for the 70% learning curve; and
(b) Use the formula to calculate the estimated direct costs and sales price of the ten units of product.

[Ans.: 2033.59 per unit**]**

Question 16: A company has 10 direct workers, who work for 25 days a month of 8 hours per day. The estimated down time is 25% of the total available time. The company received an order for a new product. The first unit of the new product requires 40 direct labour hours to manufacture the product. The company expects 80% (index is -0.322) learning curve for this type of work. The company uses standard absorption costing and the cost data are as under:

Direct materials	₹60 per unit

Direct labour	₹6 per direct labour hour
Variable overheads	Re. 1 per direct labour hour
Fixed overheads	₹7500 per month

Required:

(i) Calculate the cost per unit of the first order of 30 units.

(ii) If the company receives a repeat order for 20 units, what price to be quoted to yield a profit of 25% on selling price? (8 Marks) Nov/02 & CIMA(London)-Adapted

[Ans.: (i) 220.56 (ii) 212.88**]**

Question 17: ABC Ltd. has designed a new type of sailing boat, for which the cost and sales price of the first boat to be produced has been estimated as follows:

Materials	₹5000
Labour (800 hrs. X ₹5 per hour)	4000
Overhead (150% of labour cost)	6000
	15000
Profit mark-up (20%)	3000
Sales price	18000

It is planned to sell all the yachts at full cost plus 20%. An 80% learning curve is expected to apply to the production work. A customer interest in buying the yachts, but thinks ₹18000 is too high price to pay. He might want to buy 2, or even 4 of the yachts during the next six months.

He has asked the following questions:

(a) If he paid ₹18000 for the first yacht, what price would he have to pay later for the second yacht?

(b) Could ABC Ltd. quote same unit price for two yachts, if the customer ordered two at the same time?

(c) If the customer bought two yachts now at one price, what would be the price per unit for a third and a fourth yacht, if he ordered them separately later on?

(d) Could ABC Ltd. quote a single unit price for (i) four yachts; (ii) eight yachts; if they were all ordered now.

Assuming that there are no other prospective customers for the yachts, how would these questions be answered?

[Ans.: (a) 13200 (b) 15600 (c) 11760 (d) 13680 and 12144**]**

[Hint: In part (c) 2 yachts are being ordered at same time, so price for 3rd & 4th yacht will be 23520÷ 2**]**

Question 18: An electronics firm which has developed a new type of fire-alarm system has been asked to quote for a prospective contract. The customer requires separate price quotations for each of the following possible orders:

Order	Number of fire-alarm systems
First	100
Second	60
Third	40

The firm estimates the following cost per unit for the first order:

Direct materials	₹500
Direct labour	
Deptt. A (Highly automatic) 20 hours at ₹10 per hour	
Deptt. B (Skilled labour) 40 hours at ₹15 per hour	
Variable overheads	20% of direct labour
Fixed overheads absorbed:	
Deptt. A	₹8 per hour
Deptt. B	₹5 per hour

Determine a price per unit for each of the three orders, assuming the firm uses a mark up of 25% on total costs and allows for an 80% learning curve. Extract from 80% Learning curve table:

X	1.0	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Y%	100.0	91.7	89.5	87.6	86.1	84.4	83.0	81.5	80.0

X represents the cumulative total volume produced to date expressed as a multiple of the initial order. Y is the learning curve factor, for a given X value, expressed as a percentage of the cost of the initial order.

(11 Marks) May/10-N.C., (8 Marks) May/05, ICWA Nov/05 & CIMA – Adapted

[Ans.: (i) Price (P.V) of 1st order (100 units) = 2275 (ii) (P.V) of 2nd order (60 units) = 1848.64 (iii) (P.V) of 3rd order (40 units) = 1764.40**]**

Question 19: A company has accepted an order for making 15 items of a specialized machine at a price of ₹4 lacs each. The delivery is to be completed within 4 months. The company works 23 days a month and the normal direct wages per day amounts to ₹10000. However, in case of need, the company can work overtime up to 8 days during the said period at double the normal rate of wages. Overheads amount to ₹12000 per normal working day but no overheads are charged on overtime working days. The material cost is ₹240000 per machine. The company has estimated that it will take 10 working days to manufacture the first machine. The company is expected to experience a learning effect of 90% (b=-0.152). The contract stipulates a penalty of ₹40000 per machine delivered beyond the schedule of 4 months.

You are required to calculate the costs and advise the company whether it is preferable to work only during normal working days and pay penalty for any delayed delivery of the machines or to work overtime to avoid paying penalty. (ICWA June, 2000)

[Ans.: Additional profit is ₹6250**]**

Question 20: A firm has developed a product for which the following standard cost estimates have been made for first batch to be manufactured in Month 1:

Standard Cost for the batch

500 labour hours @ 8 per hour	₹4000
55 units of direct materials @ ₹100 per unit	5500
Variable overhead 500 hours @ 15 per hour	7500
	17000

From experience the firm knows that labour will benefit from a learning benefit and labour time will be reduced. This is expected to approximate to an 80% learning curve and to follow the general function.

$$y = ax^b$$

Where y = average labour hours per batch
a = number of labour hours for the first batch
x = cumulative number of batches
and b = learning coefficient.

(The learning coefficient is found as follows:

$$b = \log (1 - \text{proportionate decrease}) \div \log 2$$

The coefficient for an 80% learning curve is b = -0.322.)

In addition, the growing expertise of labour is expected to improve the efficiency with which materials are used. The usage of materials is expected to approximate to a 95% learning curve and to follow the general function.

$$y = ax^b$$

Where y = average material quantity per batch
a = number of material quantity for the first batch
and x and b are explained previously.

The actual production for the first six months was as follows:

Month 1	20 batches	Month 4	24 batches
Month 2	30 batches	Month 5	33 batches
Month 3	25 batches	Month 6	28 batches

During Month 6 the following results were recorded for the last batch made:

Actual results of last batch

Labour hours	115
Direct wages	₹978
Direct materials (41 units)	3977
Variable overhead	1685

You are required:

(a) to calculate the learning coefficient for materials;
(b) to derive the Standard Cost of the last batch in Month 6;
(c) to calculate what variances have arisen in connection with the last batch;
(d) to explain what information the variances provide for management.

[Ans.: (a) -0.074 (b) ₹6021 (c) LRV = 58V, LEV = 140A, MPV = 123F, MVV = 322A, VOExp. V = 40F, VOEff. V = 262A**]**

Please go through concepts of Absorption Costing before doing Question 16 and Question 19, through Standard Costing before doing Question 20, & through Direct Costing before doing Question 14

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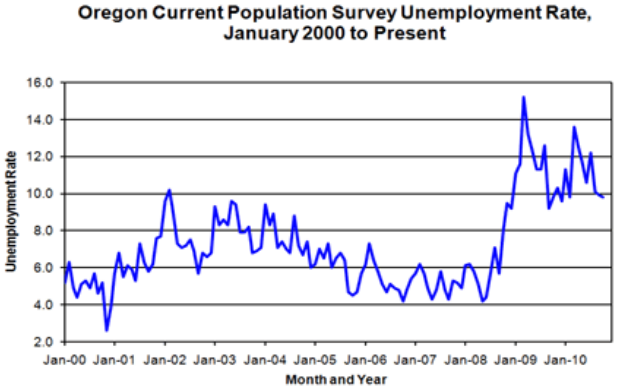
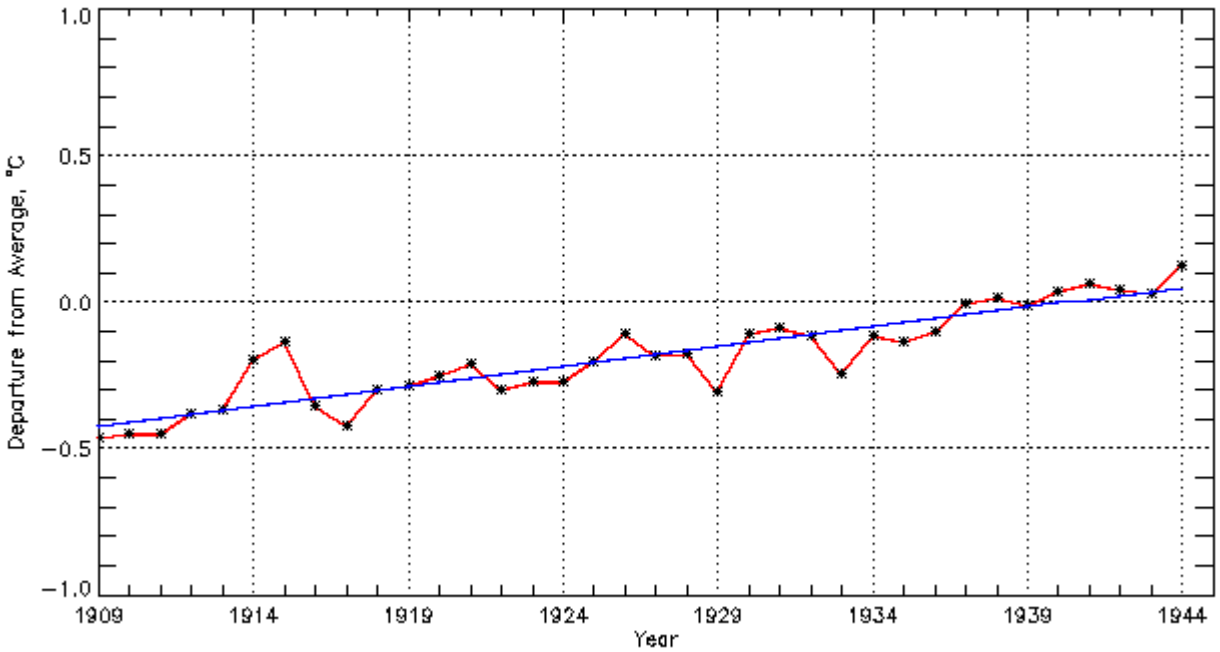


Time Series Analysis & Forecasting

Forecasting is a basic tool to help managerial decision making. Managerial decisions are seldom made in the absence of some form of forecasting. For e.g. a manager is interested in finding out his likely sales in the year 2010 or as a long-term planning in 2020 so that he could adjust his production accordingly and avoid the possibility of either unsold stocks or inadequate production to meet the demand. However, the first step in making estimates for the future consists of gathering information from the past. In this connection one usually deals with statistical data which are collected, observed or recorded at successive intervals of time. Such data are generally referred to as 'time series'. For e.g., if we observe production, sales, population, imports, exports, etc. at different point of time, say, over the last 5 or 10 years, the set of observations formed shall constitute time series.

Utilities of Time Series Analysis:

(1) It helps in evaluating current accomplishments.
(2) It helps in planning future operations.
(3) It facilitates comparison.
(4) It helps in understanding past behavior.



Various Components of Time Series:

A time series is the result of a number of movements which are caused by numerous economic, political, natural and other factors. The analysis of time series means decomposing the past data into components and

then projecting them forward. A time series typically has four components, though on occasions only one or two of these may eclipse the others.

1. Secular Trend (T) (a.k.a. Long term trend) - Trend is the general direction in which something tends to move. It is the relatively consistent movement of a variable over a long period. For e.g. a glance at the sales of a popular soft drink manufacturer is likely to reveal an increasing trend.

2. Seasonal Variations (S) - Seasonal variations are those periodic movements in business activity which occur regularly within a definite period, may be every week or month or quarter. Since these variations repeat during a period of 12 months they can be predicted fairly accurately. For example, toy manufacturers have sales increases before Christmas.

3. Cyclical Variations (C) - These are caused by business cycles or trade cycles. Cyclic fluctuations are long-term movements that represent consistently recurring rise and decline in activity. For e.g. Sales may be low due to overall subdued economic activity.

4. Irregular Variations (I) (a.k.a. Random movements) – These are residual, or erratic movements that do not have any set pattern and are usually caused by unpredictable reason, like earthquake, fire, wars, etc.

Models for Decomposition of Time Series

(a) Multiplicative Model: It is assumed that there is a multiplicative relationship between these four components i.e. it is assumed that any particular value in a series is the product of factors that can be attributed to the various components. Symbolically, $Y = T \times S \times C \times I$

(b) Additive Model: In this approach we assumes that all the components of the time series are independent of one another. For e.g. it assumes that trend has no effect on the seasonal component, no matter how high or low this value may become. Symbolically, $Y = T+S+C+I$

Note: In multiplicative model S, C, and I indexes are expressed as decimal per cents.

Question 1: What is trend? What are the various methods of fitting a straight line to a time series? (3 Marks) Nov/08

Question 2: Name the various methods of fitting a straight line to a time series and briefly explain any two of them. (5 Marks) June/09

Question 3: Identify the characteristics movement such as regular, irregular, cyclical, seasonal, long-term trend, short-term etc. of time series in the following situations:

(i) A factory delaying its production due to demolition of factory shed in earthquake.
(ii) An era of Depression in business.
(iii) The country needs more and more food grains due to constant growth of population.
(iv) Decline in death rate due to availability of proper health care facilities.
(v) A continuous increase in demand of small cars.
(vi) A demand of gold products is increasing during the festival time. (3 Marks) May/10

[Ans.: (i) Irregular (ii) Cyclical (iii) Long term trend (iv) Long term trend (v) Long term trend (vi) Seasonal**]**

Trend Projections

The various methods that can be use for determining trend are:

(a) Freehand or graphic method, (b) Semi-Average method,
(c) Moving average method, and (d) Method of least squares.

(a) Freehand or graphic method: It is the simplest method of fitting a trend line. The procedure of obtaining a straight line trend by this method is given below.

(i) Plot the time series on the graph.
(ii) Examine carefully the direction of the trend based on the plotted information (dots).
(iii) Draw the straight line which will best fit to the data according to personal judgment. The line now shows the directions of the trends.

Question 4: Fit a trend line to the following data by the freehand method:

Year	Production of steel (in million tones)	Year	Production of steel (in million tones)
1987	20	1992	25
1988	22	1993	23
1989	24	1994	26
1990	21	1995	25
1991	23		

(b) Semi-Average Method: Here we break the series into two parts, take the average (i.e. arithmetic mean) of each part. We thus get two points. Each point is plotted at the mid-point of the class interval covered by the respective part and then connect the points by a straight line.

Note: 1. In case total no. of years are in odd no., then while breaking the series in two parts we will leave out the middle year, although while preparing graph all years will be taken on x-axis (including middle year).
2. In case the parts formed on breaking the series in two parts are in even no., then while centering the average, 1st July of middle 2 year will be taken. E.g. two parts formed are [1999, 2000, 2001 & 2002] and [2003, 2004, 2005 & 2006], in such case average will be centered corresponding to 1st July 2000 & 1st July 2004.

Question 5: Fit a trend line to the following data by the method of semi-average:

Year	Sales of Firm A (Thousand Units)	Year	Sales of Firm A (Thousand Units)
1989	102	1993	108
1990	105	1994	116
1991	114	1995	112
1992	110		

[Hint: Avg. of Ist 3 years is 107 & Avg. of last 3 years is 112**]**

Question 6: Fit a trend line to the following data by the method of semi-average:

Production from 1999-2006:

Year	X	Y	Year	X	Y
1999	0	10	2003	4	20
2000	1	12	2004	5	25
2001	2	18	2005	6	23
2002	3	20	2006	7	32

[Hint: Avg. of Ist 4 years is 15 & Avg. of last 4 years is 25**]**

Question 7: Fit a trend line by the method of semi-averages to the data given below. Estimate the sales for 1996. If the actual sale for that year is Rs. 520 lakhs, account for the difference between the two figures.

Year	Sales (Rs. Lakhs)	Year	Sales (Rs. Lakhs)
1988	412	1992	479
1989	438	1993	482
1990	444	1994	490
1991	454	1995	500

[Hint: Avg. of Ist 4 years is 437 & Avg. of last 4 years is 487.75**]**

(c) Moving average method: This method consists of taking arithmetic mean of the values of a certain time span (no. of years or any other period) and placing it at the centre of the time span. We then repeat the procedure by dropping the first yearly figures of the values and adding the figure of the next which we had previously not added; thus we move the span and its centre move by one year; then we compute and place a new average. We continue this until we exhaust the series.

Since the moving average method is most commonly applied to data which are characterized by cyclical movements, it is necessary to select a period for moving average which coincides with the length of the cycle such as 3-yearly moving average, 5-yearly moving average, etc.

Note: Method of centering the moving averages (for even period):- If we are calculating 4-yearly moving average we will first take four-yearly totals and of these totals we will again take 2-yearly totals and divide these totals by 8. Alternatively, we can first take four-yearly totals and of these totals, calculate 4-yearly moving average by dividing by 4 & then we will again take 2 yearly average out of this 4-yearly moving average.

Question 8: (a) Calculate the 3-yearly moving averages of the production figures given below to determine the trend values. Find the short-term fluctuations.

Year	Production (In Met. Tons)	Year	Production (In Met. Tons)
1981	15	1989	63
1982	21	1990	70
1983	30	1991	74
1984	36	1992	82
1985	42	1993	90
1986	46	1994	95
1987	50	1995	102
1988	56		

(b) Construct 5-yearly moving averages of the number of students studying in a college shown below:

Year	No. of Students	Year	No. of Students
1987	332	1992	405
1988	317	1993	410
1989	357	1994	427
1990	392	1995	405
1991	402	1996	431

[Ans.: (a) Short term fluctuations:

Years	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994
Multiplicative	95.45	103.45	100.00	101.62	100.00	98.68	99.41	100.00	101.45	98.23	100.00	101.12	99.3
Additive	-1	1	0	0.67	0	-0.67	-0.33	0	1	-1.33	0	1	-0.67

(b) Years	1989	1990	1991	1992	1993	1994
5-Year moving avg:	360.0	374.6	393.2	407.2	409.8	415.6**]**

ICAI's study material has used multiplicative model in some questions & additive model in other questions while eliminating trend & calculating seasonal indices, so we can adopt any of the two models if question is silent, after giving prompt assumption note clarifying the relationship model used. (Clarified by ICAI via mail - posted on group on 13th Jan'10)

Question 9: Find out the three year moving averages starting from 1989.

Year:	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Sales:	10	15	20	25	15	12	15	24	15	24	15	24

[Ans.: Years	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
3 Year moving avg:	15	20	20	17	14	17	18	20	18	21**]**

Question 10: Calculate 5-yearly and 7-yearly moving averages for the following data of the number of commercial and industrial failures in a country during 1980-1995.

Year	No. Of Failures	Year	No. Of Failures
1980	23	1988	9
1981	26	1989	13
1982	28	1990	11
1983	32	1991	14
1984	20	1992	12
1985	12	1993	9
1986	12	1994	3
1987	10	1995	1

Also plot the actual and trend values on a graph. (RTP-Nov/08)

[Ans.: Years	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993
5 Year moving avg:	25.8	23.6	20.8	17.2	12.6	11.2	11	11.4	11.8	11.8	9.8	7.8
7 Year moving avg:	-	21.9	20.0	17.6	15.4	12.4	11.6	11.6	11.1	10.1	9.0	-**]**

Question 11: Estimate the trend values using the data given below by taking a four-yearly moving average and eliminate the trend:

Year	Value	Year	Value
1983	24.1	1990	45.3
1984	25.1	1991	39.3
1985	27.3	1992	41.3
1986	28.3	1993	42.2
1987	28.1	1994	46.4
1988	29.1	1995	46.6
1989	30.1	1996	49.2

[Ans.: Years	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994
Multiplicative	102.25	102.17	98.42	93.78	87.12	120.86	97.01	97.76	97.66	102.86
Additive	0.60	0.60	-0.45	-1.93	-4.45	7.82	-1.21	-0.86	-1.01	1.29**]**

Question 12: From the following data calculate 3-yearly, 5-yearly and 7-yearly moving averages and plot data on the graph:

Year	**1981**	**1982**	**1983**	**1984**	**1985**
Cyclical Fluctuations	+2	+1	0	-2	-1

Year	**1986**	**1987**	**1988**	**1989**	**1990**
Cyclical Fluctuations	+2	+1	0	-2	-1

Year	**1991**	**1992**	**1993**	**1994**	**1995**
Cyclical Fluctuations	+2	+1	0	-2	-1

[Ans.: Years	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994
3 Year moving avg:	1.00	-0.33	-1.00	-0.33	0.67	1.00	-0.33	-1.00	-0.33	0.67	1.00	-0.33	-1.00
5 Year moving avg:	-	0	0	0	0	0	0	0	0	0	0	0	-
7 Year moving avg:	-	-	0.43	0.14	-0.28	-0.43	-0.14	-0.43	0.14	-0.28	-0.43	-	-**]**

d) Method of least squares: (i) Linear Trend: The trend line drawn by this method is the line which is an average of the movement through time of the original data. The following regression equation for a straight line can be used to express the relationship:

$$Y_c = a + bX$$

We will solve following equations to yield the values of parameters a and b of the above equation.

$$\Sigma Y = Na + b\,\Sigma X$$
$$\Sigma XY = a\,\Sigma X + b\,\Sigma X^2$$

- Y_c = the trend value (which is to be predicted)
- a = the Y-axis intercept
- b = slope of trend line
- X = the independent variable, the time
- ΣY = summation of value of dependent variable (the variable whose values are to be forecasted)
- N = No. of data points
- ΣX = summation of value of the independent variable (time in this case)
- ΣXY = summation of products of the X and corresponding values
- ΣX^2 = summation of the square of values of the independent variable

Question 13:

Year	1996	1997	1998	1999	2000
Y	270	285	295	315	300

Fit a linear trend to these data using method of least squares:
[Ans.: Y_c = 299 +15x (origin: year 1998) (x = 1- year unit)**]**
Question 14: Below are given the figures of production(in thousand quintals) of a sugar factory:

Year	1976	1977	1978	1979	1980	1981	1982
Production (In '000 qtl.)	80	90	92	83	94	99	92

(i) Fit a straight line trend to these figures.
(ii) Plot these figures on graph and show the trend line.
(iii) Eliminate the trend.

[Ans.: (i) Y_c = 90 +2x (origin: year 1979) (x = 1- year unit)

(iii) Years	1976	1977	1978	1979	1980	1981	1982
Multiplicative	95.24	104.65	104.54	92.22	102.17	105.32	95.83
Additive	-4	4	4	-7	2	5	-4**]**

Question 15: Below are given the figures of production (in thousand quintals) of a sugar factory.

Year	Production (Thousand Quintals)	Year	Production (Thousand Quintals)
1975	77	1980	91
1977	88	1981	98
1978	94	1984	90
1979	85		

(i) Fit a straight line by the 'least squares' method and tabulate the trend values.
(ii) Eliminate the trend using additive model. What components of the time series are thus left over?
(iii) What is the monthly increase in the production of sugar? (RTP-Nov/08-Adapted)

[Ans.: (i) Y_c = 88.803 + 1.38x (origin: year 1979) (x = 1- year unit); (ii) After eliminating the trend we are left with cyclical & irregular variations. The seasonal variations will be absent as the data given is annual; (iii) The monthly increase in the production of sugar is b÷12 i.e. 1.38÷12 = 0.115 thousand qtl**]**

Question 16: Fit a straight line trend by the method of least squares to the following data. Assuming that the same rate of change continues, what would be the predicted earnings for the year 1988?

Year	1979	1980	1981	1982	1983	1984	1985	1986
Earnings (Rs. In Lakhs)	38	40	65	72	69	60	87	95

(RTP-June/09-Adapted)

[Ans.: (i) Y_c = 62.0833 + 7.3333x (origin: year 1982) (x = 1- year unit); *Alternatively,* Y_c = 65.75 + 7.3333x (Origin year: Mid of 1982-83) (Deviations from 1982.5); (ii) 106.087**]**

Question 17:

Year:	1985	1986	1987	1988	1989	1990
Y	15	14	18	20	17	24

Fit a trend to these data using method of least squares.

[Ans.: Y_c = 17.2 + 1.6x (origin: year 1987) (x = 1- year unit); *Alternatively,* Y_c = 18 + 1.6x (Origin year: Mid of 1987-88) (Deviations from 1987.5)**]**

(ii) Non-Linear trend (a.k.a. Higher degree polynomial trends): When we plot original time series data on a graph preparatory to the analysis we may find that a curved line is more appropriate to the data. Then we draw a non-linear trend. The simple example of the non-linear trend is the second degree parabola, the equation of which is written in the form:

$$Y_c = a + bX + cX^2$$

When numerical values of a, b, and c have been derived, the trend value for any year may be computed by substituting in the equation the value of X for that year. The value of a, b and c can be determined by solving the following three normal equations simultaneously:

$$\Sigma Y = Na + b\,\Sigma X + c\,\Sigma X^2$$
$$\Sigma XY = a\,\Sigma X + b\,\Sigma X^2 + c\,\Sigma X^3$$
$$\Sigma X^2Y = a\,\Sigma X^2 + b\,\Sigma X^3 + c\,\Sigma X^4$$

Note: If trend equations are asked to be converted to monthly values from annual value, we will divide 'a' by 12 and 'b' by 144 (i.e. 12×12) & in non-linear trend equation, we will divide 'a' by 12 and 'b' by 144 (i.e. 12×12) 'c' by 1728 (i.e. 12×12×12).

Question 18: The prices of a commodity during 1990-95 are given below. Fit a parabola $Y=a + bX + cX^2$ to these data. Estimate the price of the commodity for the year 1996.

Year	Prices	Year	Prices
1990	**100**	**1993**	**140**
1991	**107**	**1994**	**181**
1992	**128**	**1995**	**192**

Also plot the actual and trend values on the graph.

[Ans.: $Y_c = 126.657 + 18.042x + 1.786x^2$ (origin: year 1992) (x = 1- year unit); Likely price of commodity for the year 1996 is Rs. 227.401**]**

Question 19: Fit quadratic trend:

Year	1995	1996	1997	1998	1999
Production of cars('0000)	2	4	8	14	22

[Ans.: $Y_c = 8 + 5x + x^2$ origin: year 1997) (x = 1- year unit)**]**

Question 20: The trend of the annual sales of ABC Co. Ltd. is described by the following equation:

Y_c = 30 + 3.6 X (origin 1981, X unit= 1 year, Y unit= annual sales)

Convert the equation on monthly basis.

[Ans.: Y_c = 2.5 + 0.025x**]**

Detrending: The process of eliminating the trend is referred to as detrending and the trend itself can be represented either as a straight line or some type of smooth curve.

What is left after elimination of trend is short term variation, expressed as:

Under multiplicative model, $\frac{T \times S \times C \times I}{T} = S \times C \times I$

And, Under additive model, [T + S + C + I] -T = S + C + I.

Calculation of Seasonal Variations

To obtain a statistical description of pattern of seasonal variation it will be desirable to first free the data from the effects of trend, cycles & irregular variation. Once the other components have been eliminated, we can study seasonal variations, which give a clear idea about the relative position of each season on the basis of which it is possible to plan for the season.
There are many techniques available for computing an index of seasonal variation; following are some of the most popularly used:

1. Method of Simple Averages (Weekly, Monthly or Quarterly).
2. Ratio-to-Trend method (a.k.a. Percentage-to-Trend method).
3. Ratio-to-Moving Average method (a.k.a. Percentage of Moving Average method).
4. Link Relative Method (a.k.a. Pearson's method)

1. Method of Simple Averages: Steps for calculating index are:

(a) Arrange data by months, quarters, etc. & find monthly/quarterly totals.
(b) Obtain yearly average for every month/quarter by dividing totals by the no. of years.
(c) Calculate average of monthly/quarterly averages by dividing the total of monthly/quarterly averages by 12/4.
(d) Calculate Seasonal Indices: Multiplicative model:-

$$\text{Seasonal Index} = \frac{\text{Monthly or quarterly average}}{\text{Grand average of the months or quarters}} \times 100$$

Additive model:- Monthly or quarterly average – Grand average of the months or quarters

(e) Seasonal Index Adjustment: Multiplicative model:- If total of seasonal indices, in case of monthly/quarterly data is not equal to 1200/400, each seasonal index is multiplied by the factor:

$$\frac{1200/400}{\text{Sum of seasonal indices}}$$

Additive model:- If total of seasonal indices, in case of monthly/quarterly data is not equal to 0, each seasonal index is subtracted by the factor:

$$\frac{\text{Total of Seasonal Indices}}{\text{No. of months/quarters}}$$

Note: In Method of Simple averages for calculating seasonal index, we assume that trend is absent.

Question 21: Assuming that trend is absent, determine if there is any seasonality in the data given below:

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
1991	3.7	4.1	3.3	3.5
1992	3.7	3.9	3.6	3.6
1993	4.0	4.1	3.3	3.1
1994	3.3	4.4	4.0	4.0

What are the seasonal indices for various quarters?

[Ans.: Quarters	1st	2nd	3rd	4th
Multiplicative	98.66	110.74	95.30	95.30
Additive	-0.050	0.400	-0.175	-0.175**]**

Question 22: Consumption of monthly electric power in millions of kwh for street lighting in a big city during 1990-94 is given below:

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	August	Sept.	Oct.	Nov.	Dec.
1990	318	281	278	250	231	216	223	245	269	302	325	347
1991	342	309	299	268	249	236	242	262	288	321	342	364
1992	367	328	320	287	269	251	259	284	309	345	367	394
1993	392	349	342	311	290	273	282	305	328	364	389	417
1994	420	378	370	334	314	296	305	330	356	396	422	452

Find out seasonal variation by the method of monthly averages.

[Ans.: Months	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Multiplicative	116.14	103.88	101.61	91.57	85.44	80.33	82.79	90.05	97.88	109.13	116.52	124.66
Additive	51.1	12.3	5.1	-26.7	-46.1	-62.3	-54.5	-31.5	-6.7	28.9	52.3	78.1**]**

Question 23: Using the method of monthly averages determine the monthly indices for the following data of production of a commodity for the year 1979,1980,1981.

Months	1979	1980	1981
	Production in lacs of tons		
January	12	15	16
February	11	14	15
March	10	13	14
April	14	16	16
May	15	16	15
June	15	15	17
July	16	17	16
August	13	12	13
September	11	13	10
October	10	12	10
November	12	13	11
December	15	14	15

[Ans.: Months	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Multiplicative	104.9	97.5	90.2	112.2	112.2	114.6	119.5	92.6	82.9	78.0	87.8	107.3
Additive	0.67	-0.33	-1.33	1.67	1.67	2	2.67	-1	-2.33	-3	-1.67	1

2. Ratio-to-Trend Method: Steps for calculating index are:
(a) By using the method of least squares, the trend values are obtained.
(b) Multiplicative model:- The original time series is to be divided by the trend values obtained earlier. These figures are to be transformed into percentages. Now these percentage figures have three components of the time series, viz. seasonal, cyclical and irregular as the trend has been eliminated.
Additive model:- The trend value obtained should be subtracted original time series.
(c) These figures are to be averaged for each month or quarter or for any other time period in which the original data are available. This process will eliminate the effects of both cyclical and irregular movements. It may be noted that while averaging the figures, median should be preferred to the arithmetic mean. This is because the latter gives undue weightage to extreme values, which are mainly on account of seasonal swings.
(d) Seasonal Index Adjustment: Multiplicative model:- If total of seasonal indices, in case of monthly/quarterly data is not equal to 1200/400, each seasonal index is multiplied by the factor:

$$\frac{1200/400}{\text{Sum of seasonal indices}}$$

Additive model:- If total of seasonal indices, in case of monthly/quarterly data is not equal to 0, each seasonal index is subtracted by the factor:

$$\frac{\text{Total of Seasonal Indices}}{\text{No. of months/quarters}}$$

Question 24: Find seasonal variation by the ratio-to-trend method for the following data:

Year	Quarters			
	Ist	2nd	3rd	4th
1997	75	60	54	59
1998	88	65	63	80
1999	95	74	66	85
2000	100	78	73	93
2001	118	100	88	110

[Ans.: Quarters	1st	2nd	3rd	4th
Multiplicative	122.96	94.07	83.14	99.83
Additive	17.6	-4.6	-13.6	0.6**]**

Question 25: Find seasonal variation by the ratio-to-trend method from the data given below:

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
1990	30	40	36	34
1991	34	52	50	44
1992	40	58	54	48
1993	54	76	58	62
1994	80	92	86	82

[Ans.: Quarters	1st	2nd	3rd	4th
Multiplicative	92.048	117.364	102.125	88.463
Additive	-3.9	9.1	1.3	-6.5**]**

Question 26: Following data relate to sales of TULSIAN Ltd.

Year	Quarterly Sales (Rs. Lakhs)			
	Ist	2nd	3rd	4th
1997	8	16	24	32
1998	48	36	24	12
1999	48	16	32	64
2000	72	108	144	36
2001	56	28	84	112

Required: Seasonal indices by Ratio to Trend Method.

[Ans.: Quarters	1st	2nd	3rd	4th
Multiplicative	107.90725	89.65125	111.72525	90.71125**]**

3. Ratio-to-Moving Average Method: Steps for calculating index are:
(a) Calculate 12-months/4-quarters moving total for the time series
(b) Compute 12-months/4-quarters moving average by dividing each total by 12/4 & centre the moving averages, it will smooth out S × I and the remainder will be T × C. In other words, the use of the moving average gives the trend and cycle, T × C.
(c) Calculate the percentage of the actual value to the moving-average value for each month/quarter in the time series having a 12-month/4-quarter average entry. Symbolically, $\frac{T \times S \times C \times I}{T \times C} = S \times I$

(d) Collect all the percentage of actual to moving-average values & arrange them by months/quarters. These figures are to be averaged for each month or quarter. This process will eliminate the effects of irregular movements. It may be noted that while averaging the figures, median should be preferred to the arithmetic mean. This is because the latter gives undue weightage to extreme values, which are mainly on account of seasonal swings. Sometimes modified mean is used as an average for each month. The modified mean is calculated by discarding the highest and lowest values for each quarter and averaging the remaining values.

Symbolically, $\frac{S \times I}{I} = S$

(e) Seasonal Index Adjustment: If total of seasonal indices, in case of monthly/quarterly data is not equal to 1200/400, each seasonal index is multiplied by the factor:

$$\frac{1200/400}{\text{Sum of seasonal indices}}$$

Question 27: Calculate seasonal indices by the 'ratio to moving average' method from the following data:

Year	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
1992	68	62	61	63
1993	65	58	66	61
1994	68	63	63	67

[Ans.:

Quarters	1st	2nd	3rd	4th
Multiplicative	105.30	95.21	100.97	98.52
Additive	3.359375	-3.015625	0.609375	-0.953125**]**

Question 28: Apply ratio to moving average method to calculate seasonal indices from the following data:

Year	1985	1986	1987	1988
January	10	11	10	12
February	12	11	12	13
March	13	12	11	13
April	15	13	12	15
May	16	14	13	16
June	16	14	15	18
July	17	15	15	20
August	18	15	17	20
September	18	15	18	21
October	19	16	20	22
November	22	18	22	24
December	22	20	24	25

[Ans.:

Months	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Multiplicative*	70.25	74.76	75.46	83.99	88.81	96.24	103.16	106.87	109.78	116.71	133.47	140.49

*Calculation of Seasonal index is based on Median**]**

4.The Link Relative Method: This method is relatively complex as it involves the averaging of the link relatives. The use of this method to calculate seasonal indices involves the following steps:

(a) As a first step, link relatives of the seasonal figures are to be calculated. A link relative is the value of one season expressed as a percentage of the preceding season.

$$\text{Link relative} = \frac{\text{Figure for current month/quarter}}{\text{Figure for preceding month/quarter}} \times 100$$

(b) The link relatives for each month or quarter are to be averaged. For this purpose, we can use either the mean or the median. As the arithmetic mean gives undue importance to extreme values, the median is preferable.

(c) The average link relatives are now to be converted into chain relatives on the basis of the first season (month or quarter).

The formula to calculate chain relatives is:

$$\frac{\text{Link relative of the current month/quarter} \times \text{Chain relative of the preceding month/quarter}}{100}$$

As this will not add to 100 due to trend effect, some adjustment is needed.

(d) The adjusted chain relatives will be calculated by subtracting correction factor (CF) from chain relatives. This correction factor is used on the assumption that there is a linear trend. Correction factor for every month/quarter will be calculated in following way:
In monthly series, it will be
For January: **0÷12** of [(tentative) Chain relative for January-100]
For February: **1÷12** of [(tentative) Chain relative for January-100]
For March: **2÷12** of [(tentative) Chain relative for January-100], and so forth.

In quarterly series, it will be
For 1st Quarter: **0÷4** of [(tentative) Chain relative for 1st Quarter-100]
For 2nd Quarter: **1÷4** of [(tentative) Chain relative for 1st Quarter -100]
For 3rd Quarter: **2÷4** of [(tentative) Chain relative for 1st Quarter -100], and so forth.
(e)This is the final step wherein we have to ensure that the adjusted chain relatives add up to 1,200 if the data are monthly or to 400 if the data are quarterly. The figures thus arrived at are the required seasonal indices.

Question 29: Apply the method of link relatives to the following data and calculate seasonal indices.

Year	1st Quarter	2 nd Quarter	3rd Quarter	4th Quarter
1990	6.0	6.5	7.8	8.7
1991	5.4	7.9	8.4	7.3
1992	6.8	6.5	9.3	6.4
1993	7.2	5.8	7.5	8.5
1994	6.6	7.3	8.0	7.1

[Ans.: Quarters 1st 2nd 3rd 4th (RTP-June/09-Adapted)
Seasonal Indices 88.18 94.01 113.21 104.60**]**

Question 30: Calculate the seasonal indices from the following data using the link relative method.

Year	Q1	Q2	Q3	Q4
1995	65	58	56	61
1996	68	63	63	67
1997	70	59	56	52
1998	60	55	51	58

[Ans.: Quarters 1st 2nd 3rd 4th
Seasonal indices 109.33 97.78 93.89 99.00**]**

Deseasonalized Data

By deseasonalized data we mean the data which show how things would have been or would be if there were no seasonal fluctuations. Before we can identify either the trend or cyclical components of time series, we must eliminate seasonal variation. To deseasonalize a time series, we divide each of the actual values in the series by the appropriate seasonal index (expressed as a fraction of 100).

Under multiplicative model, $\frac{T \times S \times C \times I}{S} = T \times C \times I$

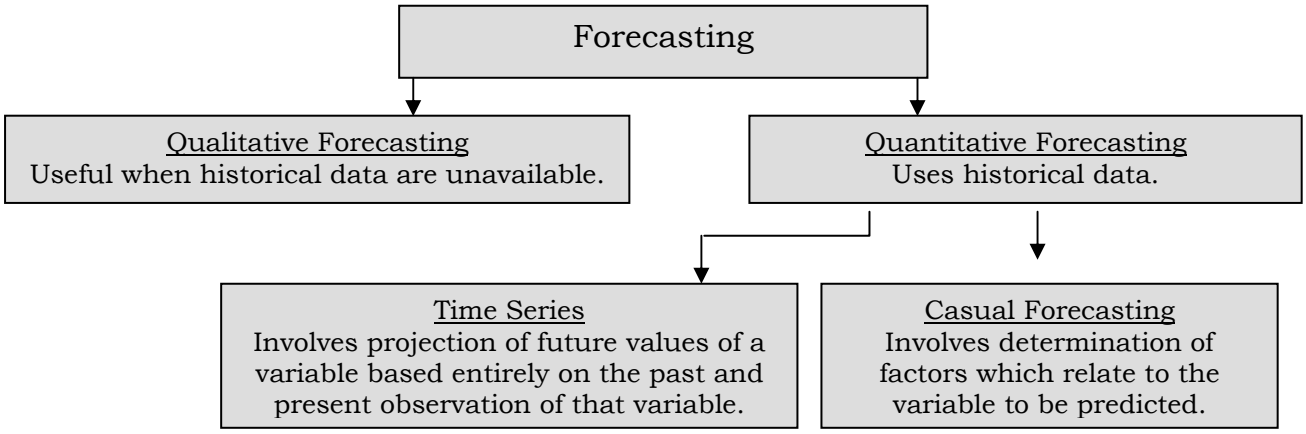
And, Under additive model, [T + S + C + I] -S = T + C + I.

Question 31: Deseasonalize the following data with the help of the seasonal data given against

Quarters	Cash Balance (`000 Rs.)	Seasonal Index
Jan - March	360	90
April – June	550	110
July – Sept.	490	100
Oct – Dec	600	120

[Ans.: 400, 500, 490, 500**]**

ICAI's study material has used multiplicative model in some questions & additive model in other questions while eliminating trend & calculating seasonal indices, so we can adopt any of the two models if question is silent, after giving prompt assumption note clarifying the relationship model used. (Clarified by ICAI via mail - posted on group on 13th Jan'10)



Various forecasting methods using time series.

(i) Mean Forecast: In this method we forecast the value of the series to be equal to the mean of the series. This method is not adequate as trend effects and the cyclical effects are not taken into account in this.

(ii) Naïve forecast: In this method, high correlation between successive pairs of values in a time series is assumed & we forecast the value, for the time period t, to-be equal to the actual value observed in the previous period t that is, time period (t–1):

(iii) Linear Trend Forecast: In this method , a linear relationship between the time and the response value has been found from the linear relationship.

$$Y_t = a + bX$$

where X will be found from the value of t and a and b are constants.

(iv) Non-linear Trend Forecast: In this method, a non-linear relationship between the time and the response value has been found again by least-squares method. Then the value, for the time period t, will be calculated from the non-linear equation . i.e.,

$$Y_t = a + bX + cX^2$$

where X-value will be calculated from the value of t.

(v) Forecasting will Exponential Smoothing: Dealt later.

Question 32: Discuss various forecasting methods using time series. (5 Marks) Nov./10

Exponential Smoothing: This is a very popular scheme to produce a smoothed Time Series. Whereas in Moving Averages the past observations are weighted equally, Exponential Smoothing assigns exponentially decreasing weights as the observation get older. In other words, recent observations are given relatively more weight in forecasting than the older observations.

Making Forecast,

$$u_t = u_{t-1} + \alpha(Y_{t-1} - u_{t-1})$$

Also, Current Forecast = Last period's forecast + α(Last period's actual value – Last period's forecasted value)

Also, Current Forecast = Last period's forecast + α(Forecasting error)

Where, α is smoothing constant & 0≤α≤1.

Choice of α: In selecting an appropriate value of the smoothing constant, the objective is to obtain the most accurate forecast. The difference between an actual value and a forecast value is called forecast error. A measure of overall error of the forecasts made is the mean of square differences between actual and forecasted values i.e. Mean Squared Error, MSE

$$\text{MSE} = \frac{\sum\left(\left|Y_{t-1} - u_{t-1}\right|\right)^2}{n} = \frac{\sum\left(\left|\text{forecast error}\right|\right)^2}{n}$$

Question 33: The actual figures for the sales of bleaching, powder for the year 2008 are shown below in the table below. The forecasts or the estimates using Exponential smoothing are required , α =0.3. Initial forecast may be taken as 29.6.

Month	Jan.	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales	33	31	34	32	37	36	34	32	41	44	44	50

[Ans.:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Forecast	29.6	30.6	30.7	31.7	31.8	33.4	34.2	34.1	33.5	35.7	38.2	39.9**]**

Question 34: The demand for a particular item during the ten months of a year is as given below. The manager is considering how well the exponential smoothing serves as an appropriate technique in forecasting the demand of this item. She is testing three values of the smoothing constant α = 0.2, α = 0.5 & α = 0.8. You are required to (a) calculate forecasted value using each of the given values, assuming the initial forecast as 208, and (b) calculate MSE for each of these series of estimates and suggest which of them is most appropriate.

Month	1	2	3	4	5	6	7	8	9	10
Sales	213	201	198	207	220	232	210	217	212	225

[Ans.:

	Months	1	2	3	4	5	6	7	8	9	10	11	MSE
α = 0.2	Forecast	208.0	209.0	207.4	205.5	205.8	208.7	213.3	212.7	213.5	213.2	215.6	109.501
α = 0.5	Forecast	208.0	210.5	205.8	201.9	204.4	212.2	222.1	216.1	216.5	214.3	219.6	111.946
α = 0.8	Forecast	208.0	212.0	203.2	199.0	205.4	217.1	229.0	213.8	216.4	212.9	222.6	120.922

Since MSE is least for α = 0.2, therefore, this smoothing constant is most preferable.**]**

Trend-adjusted Exponential smoothing:

$$u_t = u_{t-1} + \alpha(Y_t - u_{t-1})$$

$$\lambda_t = \alpha\,(u_t - u_{t-1}) + (1-\alpha)\lambda_{t-1}$$

$$u_t' = u_t + \left(\frac{1-\alpha}{\alpha} + 1\right)\lambda_t$$

Here, **u_t gives the smoothened average** at time period t λ_t gives smoothed trend & u_t' gives the necessary forecast *for the next period* at time period t.

Question 35: Exponential Smoothing for series with Trend.
Assumed : Initial Trend = -3.00, Initial Average = 750.00, α =0.1

Period	0	1	2	3	4	5	6	7	8	9	10	11	12
Sales		720	670	680	740	720	940	1020	1220	1260	1300	1190	1080

[Ans.:

Year	2	3	4	5	6	7	8	9	10	11	12
Forecast	717	704.6	696.2	701.3	701.7	746.3	800.3	886.3	967.3	1043.8	1086.3
Forecast error	47	24.6	-43.79	-18.75	-238.28	-273.71	-419.63	-373.71	-332.74	-146.23	6.3**]**

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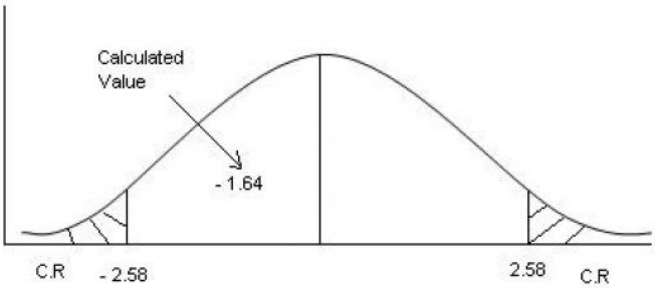


Sampling & Test of Hypothesis

Sampling

Population: All items being considered for study.
When we describe characteristics of population, they are called parameters. E.g. If mean marks in Costing & O.R. of all CA Final Students are 50, in this case 50 marks is a characteristic of the population all CA Final Students & can be call a *population parameter*.

Sample: A part or portion of the population chosen for study.
When we describe the characteristics of a sample, they are called statistics/sample statistics. E.g. If we say that mean marks in Costing & O.R. of CA Final students studying from Parag Gupta are 65, in this case 65 marks is a characteristic of the sample "Students of Parag Gupta" hence would be *sample statistics.*

Statisticians use lowercase Roman letters to denote sample statistics & Greek or capital letters for population parameters.

Key Notations used in Chapter

N	=	Population Size
n	=	Sample Size
p	=	Proportion of Population
$\bar{p}$	=	Proportion of Sample
$\bar{x}$	=	Mean of Sample
μ	=	Population Mean
z	=	Standard Score (a.k.a. standard normal variate / z transformation/test static)
σ	=	Population standard deviation
s	=	Sample standard deviation
$\sigma_{\bar{x}}$	=	Standard error of the mean
$\sigma_{\bar{p}}$	=	Standard error of proportion
$\hat{\sigma}$	=	Estimated population standard deviation [Remember, when population standard deviation is not known, we assume that sample standard deviation is equal to population standard deviation].
$\hat{\sigma}_{\bar{x}}$	=	Estimated Standard error of the mean

Sampling distribution of the mean: This is distribution of all the sample means. In statistical terminology, the sampling distribution we would obtain by taking all the samples of a given size is a theoretical sampling distribution. Statisticians have developed formulas for estimating the characteristics of these theoretical sampling distributions, making it unnecessary for us to collect large no. of samples.

Standard Error: Variability in sample means (or proportions) is measured in terms of standard errors. *Standard error* is the same basic concept as standard deviation; both represent a typical distance from the mean. S*tandard error of sample mean/proportion* is shorthand notation of "Standard deviation of the distribution of sample means/proportion".

Calculation of Standard Error:

	When population is finite **&** n÷N > 0.05	When population is infinite **or** n÷N ≤ 0.05
Standard error of the population ***mean***:		
1. When σ (population standard deviation is known)	$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \times \frac{\sqrt{N-n}}{\sqrt{N-1}}$	$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$
2. When σ (population standard deviation is **not** known) [$\hat{\sigma} = s$]	$\hat{\sigma}_{\bar{x}} = \frac{\hat{\sigma}}{\sqrt{n}} \times \frac{\sqrt{N-n}}{\sqrt{N-1}}$	$\hat{\sigma}_{\bar{x}} = \frac{\hat{\sigma}}{\sqrt{n}}$
Standard error of the population ***proportion***:		
3. When p (population proportion is known)	N.A.	$\sigma_{\bar{p}} = \sqrt{\frac{p(1-p)}{n}}$
4. When p (population proportion is **not** known)	N.A.	$\hat{\sigma}_{\bar{p}} = \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$

As per study material of ICAI & suggested answers Nov'09 (although Suggested answers of June'09 is contradicting it), whenever we are using single sample *t* test & when population is infinite **or** n÷N ≤ 0.05, formula of calculating standard error will be

- If sample standard deviation is not provided in question and we are supposed to calculate sample standard deviation for calculating standard error, then $\hat{\sigma}_{\bar{x}} = \frac{\hat{\sigma}}{\sqrt{n}}$
- If sample standard deviation is already been provided to us, then $\hat{\sigma}_{\bar{x}} = \frac{s}{\sqrt{n-1}}$

Characterizing the normal distribution

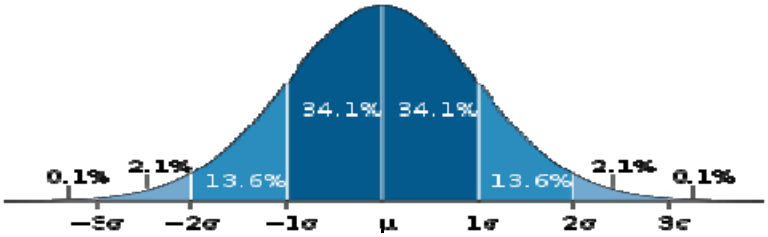
Every bell-shaped curve (normal distribution) has certain properties. You can use these properties to help determine the relative standing of any particular result in the distribution. The following is a list of properties shared by every normal distribution. These properties are explained in more detail in the following sections.

- The shape of the curve is symmetric.
- It has a bump in the middle, with tails going off to the left and right.
- The mean is directly in the middle of the distribution. The mean of the population is designated by the Greek letter μ.
- The mean and the median are the same value, due to symmetry.
- The standard deviation represents a typical (almost average) distance between the mean and all of the data. The standard deviation of the population is designated by the Greek letter σ.
- About 95% of the values are within two standard deviations of the mean.

A normal distribution is *symmetric*, meaning that if you fold it in half right down the middle, the two halves are mirror images of each other. Because its curve is symmetric, the *mean* (the balancing point) and the *median* (the point where half of the data lie on either side) are equal, and they both occur at the middle of the distribution.

Variability in a distribution is measured and marked off in terms of number of *standard deviations*
.
The **68-95-99.7 rule,** or **three sigma rule**, or empirical rule, states that for a normal distribution, almost all values lie within 3 standard deviations of the mean.

- About 68% (actually, 68.27%)of the values lie within 1 standard deviation of the mean (or between the mean minus 1 times the standard deviation, and the mean plus 1 times the standard deviation). In statistical notation, this is represented as: μ ± σ.
- About 95% (actually, 95.45%)of the values lie within 2 standard deviations of the mean (or between the mean minus 2 times the standard deviation, and the mean plus 2 times the standard deviation). The statistical notation for this is: μ ± 2σ.
- Almost all (actually, 99.73%) of the values lie within 3 standard deviations of the mean (or between the mean minus 3 times the standard deviation and the mean plus 3 times the standard deviation). Statisticians use the following notation to represent this: μ ± 3σ.



Estimates: 1. **Point Estimate:** Here we assume that sample mean $\bar{x}$ is the best estimator of the population mean μ. Further, the most frequently used estimator of the population standard deviation σ is the sample standard deviation *s.*

$$\text{Variance of Sample, } s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1}$$

[We use n-1 instead of n so that any biasness (as an estimator of population variance) would tend to be low.]

2. **Interval Estimates:** An interval estimate describes a range of values within which a population parameter is likely to lie.

Confidence Level: The probability that we associate with an interval estimate.
Confidence Interval: It is the range of the estimate we are making. Confidence Interval can be expressed as:

For Normal Distribution: $\bar{x} \pm z\sigma_{\bar{x}}$, where

$\bar{x} + z\sigma_{\bar{x}}$ is Upper limit of confidence level & $\bar{x} - z\sigma_{\bar{x}}$ is Lower limit of confidence level.

& for Student t Distribution: $\bar{x} \pm t\sigma_{\bar{x}}$, where

$\bar{x} + t\sigma_{\bar{x}}$ is Upper limit of confidence level & $\bar{x} - t\sigma_{\bar{x}}$ is Lower limit of confidence level.

Note.: 1. Confidence interval for proportions are same except that $\bar{p}$ will be used in spite of $\bar{x}$.

2. When population standard deviation is not known, $\sigma_{\bar{x}}$ will be used in spite of by $\hat{\sigma}_{\bar{x}}$

We will calculate Interval estimates using:
Student t distribution:

- Whenever sample size is 30 or less.
- Population standard deviation is not known.
- Population is normal or approximately normal

Normal Distribution: In all other cases

Question 1: The Greensboro Coliseum is considering expanding its seating capacity and needs to know both the average number of people who attend events there and the variability in this number. The following are the attendances (in thousands) at nine randomly selected sporting events. Find point estimates of the mean and the variance of the population from which the sample was drawn.

8.8 14.0 21.3 7.9 12.5 20.6 16.3 14.1 13.0

[Ans.: $\bar{x}$ = 14.278 thousand people, s^2=21.119 (thousands of people)^2**]**

Question 2: In a sample of 400 textile workers, 184 expressed extreme dissatisfaction regarding a prospective plan to modify working conditions. Because this dissatisfaction was vehement enough to allow management to interpret plan reaction as being highly undesirable, they were curious about the proportion of total workers harboring this sentiment. Give the point estimate of this proportion.

[Ans.: 0.46**]**

Question 3: From a population known to have a standard deviation of 1.4, a sample of 60 individuals is taken. The mean for this sample is found to be 6.2.

(a) Find the standard error of the mean.
(b) Establish an interval estimate around the sample mean, using one standard error of the mean.

[Ans.: (a) 0.181 (b) 6.2 ± 0.181**]**

Question 4: For a population with a known variance of 185, a sample of 64 individuals leads to 217 as an estimate of the mean.

(a) Find the standard error of the mean.
(b) Establish an interval estimate that should include the population mean 68.3 percent of the time.

[Ans.: (a) 1.70 (b) 217 ± 1.70**]**

Question 5: The manager of the Neuse River Bridge is concerned about the number of cars "running" the toll gates and is considering altering the toll-collection procedure if such alteration would be cost-effective. She randomly sampled 75 hours to determine the rate of violation. The resulting average violations per hour were 7. If the population standard deviation is known to be 0.9, estimate an interval that has a 95.5 percent chance of containing the true mean.

[Ans.: 7 ± 0.208**]**

Question 6: In what way may an estimate be less meaningful because of

(a) A high confidence level?
(b) A narrow confidence level?

[Ans.: (a) High confidence level produces wide intervals, so we sacrifice precision to gain confidence. (b) Narrow intervals result from low confidence levels, so we sacrifice confidence to gain precision.**]**

Question 7: Upon collecting a sample of 250 from a population with standard deviation of 13.7, the mean is found to be 112.4.

(a) Find a 95 percent confidence interval for the mean.
(b) Find a 99 percent confidence interval for the mean.

[Ans.: (a) 112.4 ± 1.697 (b) 112.4 ± 2.234**]**

Finite Population:

Question 8: From a population of 540, a sample of 60 individuals is taken. From this sample, the mean is found to be 6.2 and the standard deviation 1.368.

(a) Find the estimated standard error of the mean.
(b) Construct a 96 percent confidence interval for the mean.

[Ans.: (a) 0.167 (b) 6.2 ± 0.342**]**

Question 9: Joel Friedlander is a broker on the New York Stock Exchange who is curious about the time between the placement and execution of a market order. Joel sampled 45 orders and found that the mean time to execution was 24.3 minutes with a standard deviation of 3.2 minutes. Help Joel by constructing a 95 percent confidence interval for the mean time to execution.

[Ans.: 24.3 ± 0.945 minutes**]**

Question 10: Chief of Police Kathy Ackert has recently instituted a crackdown on drug dealers in her city. Since the crackdown began, 750 of the 12368 drug dealers is Rs.250000. The standard deviation of the dollar value of drugs for these 750 dealers is Rs.41000. Construct for Chief Ackert a 90 percent confidence interval for the mean dollar value of drugs possessed by the city's drug dealers.

[Ans.: Rs.250000 ± Rs.2380**]**

Proportion:

Question 11: When a sample of 70 retail executives was surveyed regarding the poor November performance of the retail industry, 65 percent believed that decreased sales were due to unseasonably warm temperatures, resulting in consumer's delaying purchase of cold-weather items.

(a) Estimate the standard error of the proportion of retail executives who blame warm weather for low sales.
(b) Find the upper and lower confidence limits for this proportion, given a confidence level equal to 0.95.

[Ans.: (a) 0.0570 (b) 0.65 ± 0.1117**]**

Question 12: Michael Jordon, a professional basketball player, shot 200 foul shots and made 174 of them.

(a) Estimate the standard error of the proportion of all foul shouts that Micheal makes.
(b) Construct a 98 percent confidence interval for the proportion of all foul shots that Micheal makes.

[Ans.: (a) 0.0238 (b) 0.87 ± 0.0555**]**

Question 13: The owner of the Home Loan Company randomly surveyed 150 of the company's 3000 accounts and determined that 60 percent were in excellent standing.

(a) Find a 95 percent confidence interval for the proportion in excellent standing.
(b) Based on part (a), what kind of interval estimate might you give for the absolute number of accounts that meet the requirement of excellence, keeping the same 95 percent confidence interval?

[Ans.: (a) 0.6 ± 0.0784 (b) 1800 ± 235.2 accounts**]**

Question 14: For the following sample sizes and confidence levels, find the appropriate *t* values for constructing confidence intervals:

(a) n=28; 95%.
(b) n=8; 98%.
(c) n=10; 95%.
(d) n=25; 99%.
(e) n=10; 99%.

[Ans.; (a) 2.052 (b) 2.998 (c) 2.262 (d) 2.797 (e) 3.250**]**

Question 15: Northern Orange County has found, much to the dismay of the county commissioners, that the population has a severe problem with dental plaque. Every year the local dental board examines a sample of patients and rates each patient's plaque buildup on a scale from 1 to 100 with 1 representing no plaque and 100 representing a great deal of plaque. This year, the board examined 21 patients and found that they had an average Plaque Rating Score (PRS) of 72 with a standard deviation of 6.2. Construct for Orange County a 98 percent confidence interval for the mean PRS for Northern Orange County.

[Ans.: 72 ± 3.5**]**

Question 16: The following sample of eight observations is from an infinite population with normal distribution:

75.3 76.4 83.2 91 80.1 77.5 84.8 81.0

a) Find the mean.
b) Estimate the population standard deviation.
c) Construct a 98 percent confidence interval for the mean.

[Ans.: (a) 81.1625 (b) 5.1517 (c) 81.1625 ± 5.4606**]**

Test of Hypothesis

As a consumer in this age of information, when you hear a claim being made (for example, "Our ice cream was the top choice of 80% of taste testers"), you basically have three options:

- Believe it automatically (or go the other way and reject it outright)
- Conduct your own test to verify or refute the claim

Believing results without question (or rejecting them out of hand) isn't wise; the only times you may want to do this are when the source has already established a good (or bad) name with you or the result simply isn't that important (after all, you can't go around checking *every* single claim that comes your way).

Doing a Hypothesis Test

A *hypothesis test* is a statistical procedure that's designed to test a claim. Typically, the claim is being made about a population parameter (***one number that characterizes the entire population***). Because parameters tend to be unknown quantities, everyone wants to make claims about what their values may be. For example, the claim that 25% (or 0.25) of all women have varicose veins is a claim about the proportion (that's the *parameter*) of all women (that's the *population*) who have varicose veins (that's the *variable*, having or not having varicose veins).

Defining what you're testing

To get more specific, the varicose vein claim is that the parameter, the population proportion (*p*), is equal to 0.25. (This claim is called the *null hypothesis*(H_o).) If you're out to test this claim, you're questioning the claim and have a hypothesis of your own (called the *research hypothesis*, or *alternative hypothesis*). You may hypothesize, for example, that the actual proportion of women who have varicose veins is lower than 0.25, based on your observations. Or, you may hypothesize that due to the popularity of high heeled shoes, the proportion may be higher than 0.25. Or, if you're simply questioning whether the actual proportion is 0.25, your alternative hypothesis is, "No, it isn't 0.25."

For example, if the claim is that the average time to make a Maggie Noodle is two minutes, the statistical shorthand notation for the null hypothesis in this case would be as follows: H_o: $\mu = 2$.
What's the alternative?

- The population parameter is *not equal to* the claimed value (H_1: $\mu \neq 2$).
- The population parameter is *greater than* the claimed value (H_1: $\mu > 2$).
- The population parameter is *less than* the claimed value (H_1: $\mu < 2$).

How do you know which hypothesis to put in H_o and which one to put in H_1/H_a? Typically, the null hypothesis says that nothing new is happening; the previous result is the same now as it was before, or the groups have the same average (their difference is equal to zero). In general, you assume that people's claims are true until proven otherwise. In general, when hypothesis testing, you set up H_o and H_a so that you believe H_o is true unless your evidence (your data and statistics) shows you otherwise.

After you've set up the hypotheses, the next step is to collect your evidence and determine whether your evidence corroborates the claim made in H_o. Remember, the claim is made about the population, but you can't test the whole population; the best you can usually do is take a unbiased and accurate sample.

This standardized version of your statistic is called a *test statistic*, and it's the main component of a hypothesis test. The general procedure for converting a statistic to a test statistic (standard score) in the case of means/proportions:

Test of Hypothesis : One-Sample Tests

Converting to a Standard Score (a.k.a. standard normal variate / z transformation / test static)

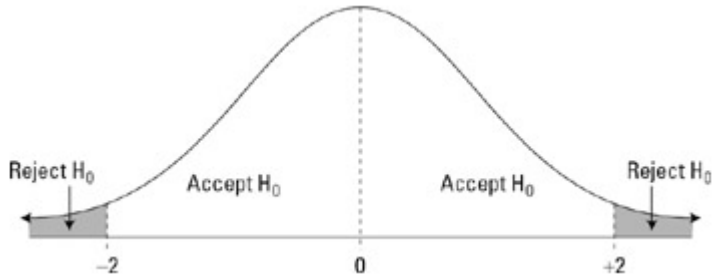
$$Z = \frac{\bar{X} - \mu_{H_0}}{\sigma_{\bar{X}}}$$

In case of proportions:

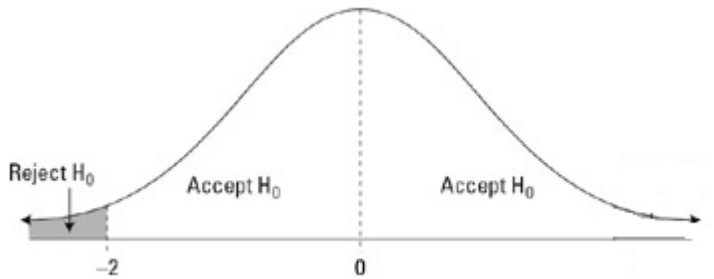
$$z = \frac{\bar{p} - p_{H_0}}{\sigma_{\bar{p}}}$$

where, $\sigma_{\bar{p}} = \sqrt{\frac{p_{H_0}(1 - p_{H_0})}{n}}$

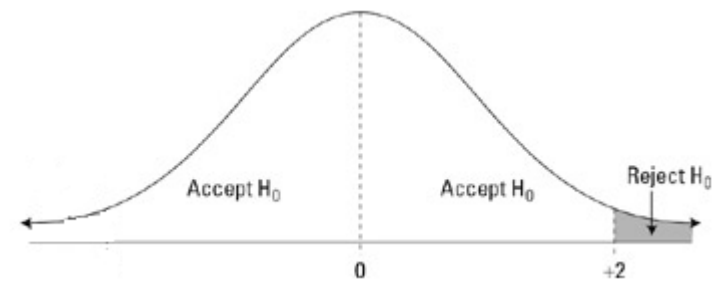
Two-tail Test



Left-tail Test



Right-tail Test



Type-1 errors - Rejecting a null hypothesis when it is true.
Type-2 errors – Accepting a null hypothesis when it is false.

Interpreting the significance level [a.k.a. *alpha level* (α)]
The purpose of hypothesis testing is not to question the computed value of the sample statistic but to make a judgment about the *difference* between the sample statistic and a hypothesized population parameter.
<u>If significance level is 5% it means that the difference between sample statistic and the hypothesized population parameter is so large(significant) that it or a larger difference would occur, on the average, only five or fewer times in every 100 samples when the hypothesized population parameter is correct.</u>

E.g. If difference observed between the sample mean $\bar{x}$ & the hypothesized population mean μ is 4.5% & our level of significance is 5%, this means that we will reject the null hypothesis since the difference between sample statistic and the hypothesized population parameter is so large that it would occur, on the average, only 4.5 times in every 100 samples (less than 5 times) when the hypothesized population parameter is correct.

While testing hypothesis, confidence interval is made around μ_{H_0} & not $\bar{X}$. It is rather tested that whether $\bar{X}$ lies within interval or not.

Summary of Five-Step Process for Hypothesis Testing under the Standardized Scale:

1. Decide whether there is a two-tailed or a one-tailed test. State your hypotheses. Select a level of significance appropriate for this decision.
2. Decide which distribution (t or z) is appropriate & find the *critical value(s)* for the chosen level of significance from appropriate table.
3. Calculate the *standard error of the sample statistic.* Use the standard error to convert the observed value of the sample statistic to a standardized value.
4. Sketch the distribution and mark the position of the standardized sample value and the critical value(s) for the test.
5. Compare the value of the standardized sample statistic with the critical value(s) for this test and interpret the result.

Question 17: Write a short note on the procedure in hypothesis testing. (5 Marks) Nov./08

Question 18: If we reject a hypothesized value because it differs from a sample statistic by more than 1.75 standard errors, what is the probability that we have rejected a hypothesis that is in fact true.?

[Ans.: 0.0802**]**

Question 19: Sports and media magnate Ned Sterner is interested in purchasing the Atlanta Stalwarts, if he can be reasonably certain that operating the team will not be too costly. He figures that average attendance would have to be about 28500 fans per game to make the purchase attractive to him. Ned randomly chooses 64 home games over the past 4 years and finds from figures reported in *Sporting Reviews* that average attendance at these games was 26100. A study he commissioned the last time he purchased a team showed that the population standard deviation for attendance at similar events had been quite stable for the past 10 years at about 6000 fans. Using 2 standard errors as the decision criterion, should Ned purchase the Stalwarts? Can you think of any reason(s) why your conclusion might not be valid?

[Ans.: $\bar{x}$=26100, $\bar{x}_L$=27000, $\bar{x}_U$=30000, so Ned should not purchase the Stalwarts. If σ has increased, the conclusion might not be valid.**]**

Question 20: An automobile manufacturer claims that a particular model gets 28 miles to the gallon. The Environmental Protection Agency, using a sample of 49 automobiles of this model, finds the sample mean to be 26.8 miles per gallon. From previous studies, the population standard deviation is known to be 5 miles per gallon. Could we reasonably expect (within 2 standard errors) that we could select such a sample if indeed the population mean is actually 28 miles per gallon?

[Ans.: $\bar{x}$=26.8 mpg, $\bar{x}_L$=26.57, $\bar{x}_U$=29.43, so such a sample is not unreasonable.**]**

Question 21: For the following cases, specify which probability distribution to use in a hypothesis test:

(a) H_o:μ = 27, H_1:μ≠27, $\bar{x}$=33, $\hat{\sigma}$ = 4, n =25

(b) H_o:μ = 98.6, H_1:μ>98.6, $\bar{x}$=99.1, σ = 1.5, n =50

(c) H_o:μ = 3.5, H_1:μ<3.5, $\bar{x}$=2.8, $\hat{\sigma}$ = 0.6, n =18

(d) H_o:μ = 382, H_1:μ≠382, $\bar{x}$=363, σ = 68, n =12

(e) H_o:μ = 57, H_1:μ>57, $\bar{x}$=65, $\hat{\sigma}$ = 12, n =42

[Ans.: (a) *t* with 24 df. (b) Normal (c) *t* with 17 df (d) Normal (e) *t* with 41df (so we use normal table)**]**

Large Sample size

Question 22: Atlas Sporting Goods has implemented a special trade promotion for its propane stove and feels that the promotion should result in a price change for the consumer. Atlas knows that the promotions should result in a price change for the consumer. Atlas knows that before the promotion began, the average retail price of the stove was Rs.44.95, with a standard deviation of Rs.5.75. Atlas samples 25 of its retailers after the promotion begins and finds the mean price for the stove is now Rs.42.95. At a 0.02 significance level, does Atlas have reason to believe that the average retail price to the consumer has decreased?

[Ans.: z = -1.74 ($\bar{x}$=Rs.42.95), z_L = -2.05 ($\bar{x}_L$=Rs.42.59), so don't reject H_o. Atlas should not believe the price has decreased**]**

Question 23: Hinton Press hypothesizes that the life of its largest web press is 14500 hours, with a known standard deviation of 2100 hours. From a sample of 25 presses, the company finds a sample mean of 13000 hours. At a 0.01 significance level, should the company conclude that the average life of the presses is less than the hypothesized 14500 hours?

[Ans.: z = -3.57 ($\bar{x}$=13000 hours), z_L = -2.33 ($\bar{x}_L$=13521 hours), so reject H_o. The average life is significantly less**]**

Question 24: The average commission charged by full-service brokerage firms on a sale of common stock is Rs.144, with a standard deviation of Rs.52. Joel Freelander has taken a random sample of 121 trades by his clients and determined that they paid an average commission of Rs.151. At a 0.10 significance level, can Joel conclude that his client's commissions are higher than the industry average?

[Ans.: z = 1.48 ($\bar{x}$=Rs.151), z_U = 1.28 ($\bar{x}_U$ =Rs.150), so reject H_o. Their commissions are significantly higher.**]**

Question 25: Prior to the 1973 oil embargo and subsequent increases in the price of crude oil, gasoline usage in the Unites States has grown at a seasonally adjusted rate of 0.57% per month, with a standard deviation of 0.10% per month. In 15 randomly chosen months between 1975 and 1985, gasoline usage grew at an average rate of only 0.33% per month. At a level 0.01 level of significance, can you conclude that the growth in the use of gasoline had decreased as a result of the embargo and its consequences?

[Ans.: z = -9.30 ($\bar{x}$=0.33 percent), z_L = -2.33 ($\bar{x}_L$ =0.51 percent), so reject H_o. The growth rate has decreased significantly.**]**

Question 26: The data-processing department at a large life insurance company has installed new color video display terminals to replace the monochrome units it previously used. The 95 operators trained to use the new machines averaged 7.2 hours before achieving satisfactory level of performance. Their sample variance was 16.2 squared hours. Long experience with operators on the old monochrome terminals showed that they averaged 8.1 hours on the machines before their performances were satisfactory. At the 0.01 significance level, should the supervisor of the department conclude that the new terminals are easier to learn to operate?

[Ans.: Z = -2.1679 ($\bar{x}$=7.2), Z_L = -2.33 ($\bar{x}_L$=7.13), so don't reject H_o. The new terminals are not easier to learn to operate.**]**

Proportions

Question 27: Grant Inc., a manufacturer of women's dress blouses, knows that its brand is carried in 19 percent of the women's clothing stores east of the Mississippi River. Grant recently sampled 85 women's clothing stores on the West Coast and found that 14.12 percent of the stores carried the brand. At the 0.04 level of significance, is there evidence that Grant has poorer distribution on the West Coast than it does east of the Mississipi?

[Ans.: z = -1.15 ($\bar{p}$=0.1412), z_L = -1.75 ($\bar{p}_L$ =0.1155), so don't reject H_o. There is no evidence that West Coast distribution is significantly worse.**]**

Question 28: Feronetics specializes in the use of gene-splicing techniques to produce new pharmaceutical compounds. It has recently developed a nasal spray containing interferon, which it believes will limit the transmission of the common cold within families. In the general population, 15.1 percent of all individuals will catch a rhinovirus-caused cold once another family member contracts such a cold. The interferon spray was tested on 180 people, one of whose family members subsequently contracted a rhinovirus-caused cold. Only 17 of the test subjects developed similar colds.

(a) At a significance level of 0.05, should Feronetics conclude that the new spray effectively reduces transmission of colds?
(b) What should it conclude at α = 0.02?
(c) On the basis of these results, do you think Feronetics should be allowed to market the new spray? Explain.

[Ans.: (a) z = -2.12 ($\bar{p}$=0.0944), z_L = -1.64 ($\bar{p}_L$ =0.1072), so reject H_o. Yes, they should conclude that transmission is reduced.

(b) z_L = -2.05 ($\overline{p}_L$ =0.0963), so the conclusion is unchanged.

(a) Not necessarily: Among other reasons, we have been given no information about potential adverse side effects of the spray.**]**

Question 29: A manufacturer claimed that at least 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipment revealed that 18 were faulty. Test this claim at a significance level of (i) 0.05 (ii) 0.01. RTP-June/09

[Ans.: (i) z = -2.596 ($\overline{p}$ =0.91), z_L = -1.64 ($\overline{p}_L$ =0.925), so reject H_o. Hence manufacturer's claim is rejected.

(ii) z = -2.596 ($\overline{p}$ =0.91), z_L = -2.33 ($\overline{p}_L$ =0.914), so reject H_o. Hence manufacturer's claim is rejected.**]**

Question 30: A ketchup manufacturer is in the process of deciding whether to produce a new extra-spicy brand. The company's marketing-research department used a national telephone survey of 6000 households and found that extra-spicy ketchup would be purchased by 335 of them. A much more extensive study made 2 years ago showed that 5 percent of the households would purchase the brand then. At a 2 percent significance level, should the company conclude that there is an increased interest in the extra-spicy flavor?

[Ans.: z = 2.07 ($\overline{p}$ =0.05583), z_U = 2.05 ($\overline{p}_U$ = 0.05577), so reject H_o. Interest has increased significantly (although just barely).**]**

Small Sample size

Question 31: Given a sample mean of 83, a sample standard deviation of 12.5, and a sample size of 22, test the hypothesis that the value of the population mean is 70, against the alternative that it is more than 70. Use the 0.025 significance level.

[Ans.: t = 4.766 ($\overline{x}$ =83), t_U = 2.080 ($\overline{x}_U$ =75.67), so reject H_o**]**

Question 32: In the past, a machine has produced pipes of diameter 50 mm. To determine whether the machine is in proper working order, a sample of 10 pipes is chosen, for which mean diameter is 53 mm and the standard deviation is 3 mm. Test the hypothesis that the machine is in proper working order, given that the critical value of the test statistic from the table is 2.26. (4 Marks) Nov./09-N.C.

[Ans.: t = 3 , t_{CRIT} = ±2.26 ,so reject H_o. Machine may not be working properly.**]**

Question 33: A factory manager contends that the mean operating life of light bulbs of his factory is 4200 hours. A customer disagrees and says it is less.

The mean operating life for a random sample of 9 bulbs is 4000 hours, with a sample standard deviation of 201 hours.

Test the hypothesis of the factory manager, given that the critical value of the test static as per the table is (-) 2.896. (6 Marks) June/09-N.C.

[Ans.: t = -2.81 ($\overline{x}$ =4000 hours), t_L = -2.896 ($\overline{x}_L$ =3994.198 hours), so accept H_o. The average life is not significantly less**]**

Question 34: A potato chips manufacturing company decided that the mean net weight per pack of its product must be 90 grams. A random sample of 16 packets yields a mean weight of 80 grams with standard deviation of 17.10 grams. Test the hypothesis that the mean of the whole universe is less than 90, use level of significance of (a) 0.05(b) 0.01. (5 Marks) Nov./10-N.C.

[Ans.: t = -2.2649 ($\overline{x}$ =80 grams); (i) t_L = -1.753 ($\overline{x}_L$ =82.26 grams), so reject H_o. The average weight is significantly less (ii) t_L = -2.602 ($\overline{x}_L$ =78.52 grams), so don't reject H_o. The average weight is not significantly less**]**

Question 35: Realtor Elaine Snyderman took a random sample of 12 homes in a prestigious suburb of Chicago and found the average appraised market value to be Rs.780000, with a standard deviation of Rs.49000. Test the hypothesis that for all homes in the area, the mean appraised value is Rs.825000, against the alternative that it is less than Rs.825000. Use the 0.05 level of significance.

[Ans.: t = -3.0456 ($\overline{x}$ =780000), t_L = -1.796 ($\overline{x}_L$ =798466), so reject H_o**]**

Question 36: A television documentary on overeating claimed that Americans are about 10 pounds overweight on average. To test this claim, eighteen randomly selected individuals were examined, and their average excess weight was found to be 12.4 pounds, with a sample standard deviation of 2.7 pounds. At a significance level of 0.01, is there any reason to doubt the validity of the claimed 10-pound value?

[Ans.: t = 3.665 ($\bar{x}$=12.4), t_{CRIT} = ±2.898 ($\bar{x}_L$=8.102, $\bar{x}_U$=11.898), so reject H_o. The claim doesn't appear to be valid.**]**

Test of Hypothesis :Two-Sample Tests

Hypothesis Testing for differences between *means*

Difference between sample means → $\bar{x}_1 - \bar{x}_2$

The *mean of the sampling distribution of the distribution of the difference between sample* means is symbolized $\mu_{\bar{x}1-\bar{x}2}$ Remember it is always equal to $\mu_{\bar{x}1} - \mu_{\bar{x}2}$

Large Sample Sizes:

$$\sigma_{\bar{x}1-\bar{x}2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Where, $\sigma_{\bar{x}1-\bar{x}2}$ =Standard error of the difference between two means

σ_1^2 = Variance of Population 1

σ_2^2 = Variance of Population 2

n_1 = Size of sample from Population 1

n_2 = Size of sample from Population 2

Similar to way done earlier, if the two population standard deviations are not know, we can estimate the standard error of the difference between two means by taking $\hat{\sigma} = s$ & formula for the estimated standard error of the difference between two means becomes

$$\hat{\sigma}_{\bar{x}1-\bar{x}2} = \sqrt{\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}}$$

Where, $\hat{\sigma}_{\bar{x}1-\bar{x}2}$ = Estimated Standard error of the difference between two means

$\hat{\sigma}_1^2$ = Estimated variance of Population 1

$\hat{\sigma}_2^2$ = Estimated variance of Population 2

$$\text{Standardized scale, } z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)_{H0}}{\hat{\sigma}_{\bar{x}1-\bar{x}2}}$$

Small Sample Sizes: (i) If sample size is 30 or less but population standard deviation is known: Use normal distribution & calculate standard error is calculated in same way as that in Large Sample sizes.

(ii) If sample size is 30 or less but population standard deviation is <u>not</u> known: Use t distribution and standard error is calculated as

$$\hat{\sigma}_{\bar{x}1-\bar{x}2} = \sqrt{\frac{n_1 \times \hat{\sigma}_1^2 + n_2 \times \hat{\sigma}_2^2}{n_1 + n_2 - 2}} \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Note: Degree of freedom will be $n_1 + n_2 - 2$

$$\text{Standardized scale, } t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)_{H0}}{\hat{\sigma}_{\bar{x}1-\bar{x}2}}$$

Hypothesis Testing for differences between *proportions*

Estimated standard error of the difference between two proportions, $\hat{\sigma}_{\bar{p}_1-\bar{p}_2} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}}$

where, Combined proportion of two samples, $\hat{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2}$

Standardized scale, $z = \dfrac{(\bar{p}_1 - \bar{p}_2) - (p_1 - p_2)_{H_0}}{\hat{\sigma}_{\bar{p}_1 - \bar{p}_2}}$

Large Sample size

Question 37: Two independent samples of observations were collected. For the first sample of 60 elements, the mean was 86 and the standard deviation 6. The second sample of 75 elements had a mean of 82 and a standard deviation of 9.

(a) Compute the estimated standard error of the difference between the two means.
(b) Using α = 0.01, test whether the two samples can reasonably be considered to have come from populations with the same mean.

[Ans.: (a) 1.2961 (b) z = 3.08 ($\bar{x}_1 - \bar{x}_2$ = 4), z_{CRIT} = ±2.57 (($\bar{x}_1 - \bar{x}_2$)$_{CRIT}$ = ± 3.331), so reject H_o**]**

Question 38: A sample of 32 money-market mutual funds was chosen on January 1, 1993, and the average annual rate of return over the past 30 days was found to be 3.23%, with a sample standard deviation of 0.51%. A year earlier, a sample of 38 money-market funds showed an average rate of return of 4.36%, with a sample standard deviation of 0.84%. Is it reasonable to conclude (at α = 0.05) that money-market interest rates declined during 1992?

[Ans.: z = 6.92 ($\bar{x}_1 - \bar{x}_2$ =1.13 percent), z_U = 1.64 (($\bar{x}_1 - \bar{x}_2$)$_U$=0.27 percent), so reject H_o**]**

Question 39: Notwithstanding the Equal Pay Act of 1963, in 1993 it still appeared that men earned more than women in similar jobs. A random sample of 38 male machine-tool operators found a mean hourly wage of Rs.11.38, with a standard deviation of Rs.1.84. A random sample of 45 female machine-tool operators found their mean wage to be Rs. 8.42, with a standard deviation of Rs.1.31. On the basis of these samples, is it reasonable to conclude (α = 0.01) that the male operators are earning over Rs.2.00 more per hour than the female operators?

[Ans.: z=2.69 ($\bar{x}_M - \bar{x}_W$ =Rs.2.96), z_U =2.33 (($\bar{x}_M - \bar{x}_W$)$_U$=Rs.2.83), so reject H_o. The male operators earn significantly more than Rs.2.00 above what the female operators earn.**]**

Small Sample size

Question 40: A consumer-research organization routinely selects several car models each year and evaluates their fuel efficiency. In this year's study of two similar subcompact models from two different automakers, the average gas mileage for 12 cars of brand A was 27.2 miles per gallon, with a standard deviation of 3.8 mpg. The nine brand B cars that were tested averaged 32.1 mpg, with a standard deviation of 4.3 mpg. At α = 0.01, should it conclude that brand B cars have higher average gas mileage than do brand A cars?

[Ans.: t = -2.628 ($\bar{x}_A - \bar{x}_B$ = -4.9 mpg), t_L= -2.539 (($\bar{x}_A - \bar{x}_B$)$_L$= -4.4734 mpg), so reject H_o. Brand B has significantly higher mileage.**]**

Question 41: To celebrate their first anniversary, Randy Nelson decided to buy a pair of diamond earring for his wife Debbie. He was shown nine pairs with marquise gems weighing approximately 2 carats per pair. Because of differences in the colors and qualities of the stones, the prices varied from set to set. The average price was Rs.2990, with a sample standard deviation of Rs.370. He also looked at six pairs with pear-shaped stones of the same 2-carat approximate weight. These earrings had an average price of Rs.3065, with a standard deviation of Rs.805. On the basis of this evidence, can Randy conclude (at a significance level of 0.05) that pear-shaped diamonds cost more, on average than marquise diamonds?

[Ans.: t = -0.2267 ($\bar{x}_1 - \bar{x}_2$ = -Rs.75), t_L= -1.771 (($\bar{x}_1 - \bar{x}_2$)$_L$= -Rs.586), so do not reject H_o. The pear-shaped stones are not significantly more expensive.**]**

Question 42: Because refunds are paid more quickly on tax returns that are filed electronically, the Commissioner of the Internal Revenue Service was wondering whether refunds due on returns filed by mail were smaller than those due on returns filed electronically. Looking only at returns claiming refunds, a sample of 17 filed by mail had an average refund of Rs.563, with a standard deviation of Rs.378. The average refund claimed on 13 electronically filed return was Rs.958, with a standard deviation of Rs.619. At α = 0.01, do these data support the commissioner's speculation?

[Ans.: t = -2.084 ($\bar{x}_M - \bar{x}_E$ = -Rs.395), t_L= -2.467 (($\bar{x}_M - \bar{x}_E$)$_L$= -Rs.467.59), so do not reject H_o. The data do not support the commissioner's speculation.**]**

Testing Differences between Means with Dependent (or paired) Samples

Question 43: The data below are a random sample of nine firms chosen from the "Digest of Earning Reports", in The Wall Street Journal on Feb 6 2008:

(a) Find the mean change in earnings per share between 2007 & 2008.
(b) Find the standard deviation of the change and the standard error of the mean.
(c) Were average earnings per share different in 2007 & 2008? Test at α = 0.02

Firm	1	2	3	4	5	6	7	8	9
2007 earnings per share	1.38	1.26	3.64	3.50	2.47	3.21	1.05	1.98	2.72
2008 earnings per share	2.48	1.50	4.59	3.06	2.11	2.80	1.59	0.92	0.47

[Ans.: t= -0.5428 ($\bar{x}$ = -0.19), $t_{CRIT} = \pm 2.896 (\bar{x}_{CRIT} = \pm 1.01)$ so do not reject H_o.**]**

Question 44: Nine computer-components dealers in major metropolitan areas were asked for their prices on two similar dot-matrix printers with standard widths and near-letter-quality fonts. The results of this survey are given below. At α = 0.05, is it reasonable to assert that, on average, the Apson printer is less expensive than HP Printer?

Dealer	1	2	3	4	5	6	7	8	9
Apson (Rs.)	350	419	385	360	405	395	389	409	375
HP (Rs.)	370	425	369	375	389	385	395	425	400

[Ans.: t = 0.981 ($\bar{x}$ = 5.11), $t_U = 1.80 (\bar{x}_U = 9.69)$ so do not reject H_o. Apson is not significantly less expensive**]**

Question 45: The sales data of an item in six shops before and after a special promotional campaign are as under:

Shops	A	B	C	D	E	F
Before Campaign	53	28	31	48	50	42
After Campaign	58	29	30	55	56	45

Can the compaign be judged to be a success?
Test at 5% level of significance using t-test. (RTP-Nov/08)

[Ans.: t = -2.782, $t_L = -2.015$ so reject H_o**]**

Question 46: Divya is a production supervisor on the disk-drive assembly line at XY Ltd. XY Ltd. recently subscribed to an easy listening music service at its factory, hoping that this would relax the workers and lead to greater productivity Divya is skeptical about this hypothesis and fears the music will be distracting, leading to lower productivity. She sampled weekly production for the same six workers before the music was installed and after it was installed. Her data are given below. At α = 0.02, has the average production changed at all?

Employee	1	2	3	4	5	6
Week without music	219	205	226	198	209	216
Week with music	235	186	240	203	221	205

[Ans.: t= 0.478 ($\bar{x}$ = 2.83), $t_{CRIT} = \pm 3.365 (\bar{x}_{CRIT} = \pm 19.95)$ so do not reject H_o.**]**

<u>Proportion</u>

Question 47: On Friday, 11 stocks in a random sample of 40 of the roughly 2500 stocks traded on the New York Stock Exchange advanced; that is, their price of their shares increased. In a sample of 60 NYSE stocks taken on Thursday, 24 advanced. At α = 0.10, can you conclude that a smaller proportion of NYSE stocks advanced on Friday than did on Thursday?

[Ans.: z = -1.283 $(\bar{p}_1 - \bar{p}_2 = -0.125), z_L = -1.28((\bar{p}_1 - \bar{p}_2 = -0.1247)$, so reject H_o, a smaller proportion advanced on Friday**]**

Question 48: A coal-fired power plant is considering two different systems for pollution abatement. The first system has reduced the emission of pollutants to acceptable levels 68 percent of the time, as determined

from 200 samples. The second, more expensive system has reduced the emission of pollutants to acceptable levels 76 percent of the time, as determined 250 air samples. If the expensive system is significantly more effective than the inexpensive system in reducing pollutants to acceptable levels, then the management of power plant will install the expensive system, Which system will be installed if management uses a significance level of 0.02 in making its decision?

[Ans.: z = -1.89 $(\bar{p}_1 - \bar{p}_2 = -0.08), z_L = -2.05((\bar{p}_1 - \bar{p}_2)_L = -0.0869)$, so don't reject H_o, install the less expensive system.**]**

Question 49: Two different areas of a large eastern city are being considered as sites for day-care centers. Of 200 households surveyed in one section, the proportion in which the mother worked full-time was 0.52. In the other section, 40 percent of the 150 households surveyed had mothers working at full-time jobs. At the 0.04 level of significance, is there a significant difference in the proportions of working mothers in the two areas of the city?

[Ans.: z = 2.23 $(\bar{p}_1 - \bar{p}_2 = 0.12), z_{CRIT} = \pm 2.05((\bar{p}_1 - \bar{p}_2)_{CRIT} = \pm 0.1105)$, so reject H_o, proportions are significantly different**]**

Chi-square and analysis of variance

Chi-square distribution is use to examine the difference among more than two sample **proportions**, ANOVA enables us to test for the significance of the differences among more than two sample **means.**

$$\text{Chi-square, } \chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

where, E_i =Expected frequency & O_i is Observed frequency

Number of degrees of freedom = (no. of rows-1)×(no. of columns-1)

Question 50: Given the following dimensions for contingency tables, how many degrees of freedom will the chi-square statistic for each have?

(a) 5 rows, 4 columns.
(b) 6 rows, 2 columns.
(c) 3 rows, 7 columns.
(d) 4 rows, 4 columns.

[Ans.: (a) 12 (b) 5 (c) 12 (d) 9**]**

Question 51: A brand manager is concerned that her brand's share may be unevenly distributed throughout the country. In a survey in which the country was divided into four geographic regions, a random sampling of 100 consumers in each region was surveyed, with the following results:

	Region				
	NE	**NW**	**SE**	**SW**	**TOTAL**
Purchase the brand	40	55	45	50	190
Do not purchase	60	45	55	50	210
Total	100	100	100	100	400

Develop a table of observed and expected frequencies for this problem.

(a) Calculate the sample χ^2 value.
(b) State the null and alternative hypothesis.
(c) If the level of significance is 0.05, should the null hypothesis be rejected?

[Ans.: (a) χ^2 = 5.012 (b) H_0: region and purchasing are independent. H_1: region and purchasing are dependent. (c) χ^2_U =7.815**]**

Question 52: To see if silicon chip sales are independent of where the U.S. economy is in the business cycle, data have been collected on the weekly sales of Zippy Chippy, a Silicon Valley firm, and on whether the U.S.

economy was rising to a cycle peak, at a cycle peak, falling to a cycle trough, or at cycle trough. The results are:

	WEEKLY CHIP SALES			
Economy	**High**	**Medium**	**Low**	**TOTAL**
At peak	20	7	3	30
At trough	30	40	30	100
Rising	20	8	2	30
Falling	30	5	5	40
TOTAL	100	60	40	200

Calculate a table of observed and expected frequencies for this problem.

(a) State the null and alternative hypothesis.

(b) Calculate the sample χ^2 value.

(c) At the 0.10 significance level, what is your conclusion?

[Ans.: (a) H_0: sales and economy are independent. H_1: sales and economy are dependent. (b) χ^2 = 34.597 (c) χ^2_U =10.645, so we reject H_0**]**

Question 53: A newspaper publisher, trying to pinpoint his market's characteristics, wondered whether newspaper readership in the community is related to reader's educational achievement. A survey questioned adults in the area on their level of education and their frequency of readership. The results are shown in the following table.

	LEVEL OF EDUCATION ACHIEVEMENT				
FREQUENCY OF READERSHIP	**Professional or postgraduate**	**College graduate**	**High school grad**	**Did not complete high school**	**TOTAL**
Never	10	17	11	21	59
Sometimes	12	23	8	5	48
Morning or evening	35	38	16	7	96
Both editions	28	19	6	13	66
TOTAL	85	97	41	46	269

At the 0.10 significance level, does the frequency of newspaper readership in the community differ according to the reader's level of education?

[Ans.: χ^2 = 32.855, χ^2_U =14.684, so we reject H_0 Different levels of education do correspond to different frequencies of readership.**]**

Question 54: The contingency table below summarize the results obtained in a study conducted by a research organization with respect to the performance of four competing brands of tooth paste among the users

	Brand A	**Brand B**	**Brand C**	**Brand D**	**Total**
No. of Cavities	9	13	17	11	50
One of five	63	70	85	82	300
More than five	28	37	48	37	150
Total	100	120	150	130	500

Test the hypothesis that incidence of cavities is independent of the brand of the tooth paste used. Use level of significance 1% and 5%.

[Ans.: χ^2 = 1.911, χ^2_U =12.59, so we accept H_0. Hence, incidence of cavities is independent of the brand of the tooth paste used.**]**

Question 55: Given below in the contingency table for production is three shifts and the number of defective good turn out- Find the value of χ^2. Is it possible that the number defective goods depends on the shifts run by them, No of Shifts:

Shift	I Week	II Week	III Week	Total
I	15	5	20	40
II	20	10	20	50
III	25	15	20	60
	60	30	60	150

[Ans.: χ^2 = 3.647, χ^2_U =9.488, so we accept H_0. Hence, the number of defective does not depends on the shift run by the factory**]** (RTP-June/09)

Analysis of Variance (ANOVA)

One Way Classification:

Sum of Squares for Columns, SSC = $\sum n(\bar{x} - \bar{\bar{x}})^2$

Mean Sum of Squares for Columns, $MSC = \dfrac{SSC}{c-1}$

Sum of Squares due to error, SSE = $\sum_i \sum_j (x_{ij} - \bar{x}_j)^2$

Mean Sum of Squares due to error, $MSE = \dfrac{SSE}{t-c}$

Variance Ratio, $F = \dfrac{MSC}{MSE}$

Degree of Freedom between Columns = c-1
Degree of Freedom between Rows = t-c

where, $\bar{\bar{x}}$ = grand mean i.e. mean of means,
c = total no. of columns
r = total no. of rows
i = no. of row
j = no. of column

Question 56: The three samples below have been obtained from national populations with equal variances.

8	7	12
10	5	9
7	10	13
14	9	12
11	9	14

Test the hypothesis at 5% level that the population means are equal.
(The table value of F at 5% level of significance for v_1=2 and v_2=12 is 3.88)

[Ans.: F = 4, The table values of F at 5% level of significance for (2, 12) degrees of freedom is 3.88. The calculated value of F is more than the table value of F. Hence, The null hypotheses is rejected. We, therefore conclude that the population means are not equal.**]**

Question 57: Below are given the yield (in kg.) per acre for 5 trial plots of 4 varieties of treatment. Carry out an analysis of variance and state conclusion

	Treatment			
Plot no.	**1**	**2**	**3**	**4**
1	42	48	68	80
2	50	66	52	94
3	62	68	76	78

4	34	78	64	82
5	52	70	70	66

[Ans.: F = 8.3, The table values of F at 5% level of significance for (3, 16) degrees of freedom is 3.24. The calculated value of F is more than the table value of F. Hence, The null hypotheses is rejected. Hence, The treatment does not have same effect.**]** (RTP-Nov/08)
[Note: ICAI has assumed that one way classification should be used in above question instead of manifold classification**]**

Question 58: The manager of an assembly line in a clock manufacturing plant decided to study how different speeds of the conveyor belt affect the rate of defective units produced in an 8-hour shift. To examine this, he ran the belt at four different speeds for five 8-hour shifts each and measured the number of defective units found at the end of each shift. The results of the study follow:

	Defective Units per Shift		
Speed 1	**Speed 2**	**Speed 3**	**Speed 4**
37	27	32	35
35	32	36	27
38	32	33	33
36	34	34	31
34	30	40	29

(a) Calculate the mean number of defective units, $\bar{x}$, for each speed; then determine the grand mean, $\bar{\bar{x}}$.
(b) Estimate the population variance (the between column variance) (i.e. MSC).
(c) Calculate the variances within the samples and estimate the population variance based upon these variances (the within column variance) (i.e. MSE).
(d) Calculate the F ratio. At the 0.05 level of significance, do the four conveyor-belt speeds produce the same mean rate of defective clocks per shift?

[Ans.: (a) $\bar{x}_j$ =36.31, 31,35,31; $\bar{\bar{x}}$ = 33.25 (b) 34.5833 (c) 7.375 (d) F= 4.69, F_U= 3.24, so we reject H_o**]**

Manifold Classification:

Correction Factor, $C = \frac{T^2}{r \times c}$

Sum of Squares between columns, $SSC = \sum \frac{(\sum x_j)^2}{n_j} - C$

Sum of Squares between rows, $SSR = \sum \frac{(\sum x_i)^2}{n_i} - C$

Total Sum of Squares, $SST = \sum_{i=1}^{r}\sum_{j=1}^{c} x_{ij}^2 - C$

Sum of Squares due to Error, SSE = SST-{SSC+SSR}
Degree of Freedom between Columns = c-1
Degree of Freedom between Rows = r-1
Degree of Freedom for total = rc-1
Degree of Freedom between errors = [rc-1]-[(c-1)+(r-1)]

where, T = grand total
c = total no. of columns
r = total no. of rows
i = no. of row
j = no. of column

Question 59: A farmer applies three types of fertilizers on 4 separate plots. The figures on yield per acre are tabulated below:

Fertilizers/Plots	Yield A	B	C	D	Total
Nitrogen	6	4	8	6	24
Potash	7	6	6	9	28
Phosphates	8	5	10	9	32
Total	21	15	24	24	84

Find out if the plots are materially different in fertility as also if the three fertilizers make any material difference in yields.

[Ans.: **ANOVA TABLE**

Sources of variation	d.f	S.S	MSS	Variance Ratio (F)
Rows (Fertilizers)	2	8	MSR=4	2.4
Columns (Plots)	3	18	MSC=6	3.6
Errors	6	10	MSE=1.667	
Total	11	36		

$F_{(0.05, 3, 6)}$ is 4.76 & $F_{(0.05, 2, 6)}$ is 5.14. The calculated value of F is less than the table value of F. Hence, The null hypotheses is accepted.**]**

Question 60: For the following data representing the number of units of production per day turned out by five workers using from machines, set-up the ANOVA table (Assumed Origin at 20).

Workers	Machine Type A	B	C	D
1.	4	-2	7	-4
2.	6	0	12	3
3.	-6	-4	4	-8
4.	3	-2	6	-7
5.	-2	2	9	-1

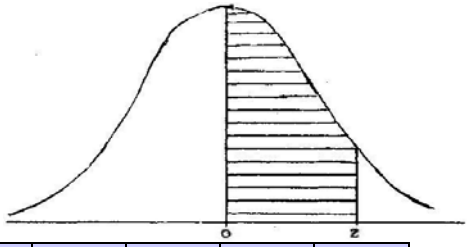
(RTP-June/09)

[Ans.: **ANOVA TABLE**

Sources of variation	d.f	S.S	MSS	Variance Ratio (F)
Rows (Workmen)	4	161.5	40.38	6.576
Columns (Machine)	3	33.8	112.93	18.39
Errors	12	73.7	6.14	
Total	19	574		

$F_{(0.05, 4, 12)}$ is 3.259 & $F_{(0.05, 3, 12)}$ is 3.49. The calculated value of F is more than the table value of F. Hence, The null hypotheses is rejected.**]**

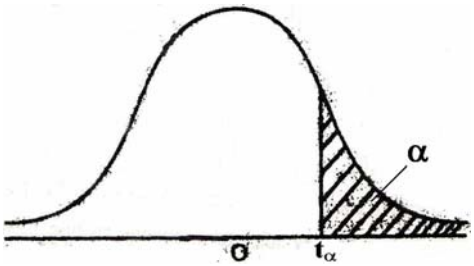
Table for ***Areas Under the Standard Normal Curve***
from 0 to Z (Type II)
[P (0 ≤ X ≤ x) = n (0 ≤ Z ≤ z)]



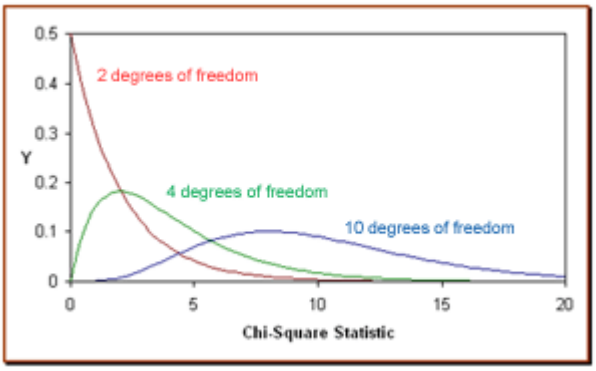
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3304	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999

Student's *t* Distribution

The following table provides the values of t_α that correspond to a given upper-tail area α and a specified number of degree of freedom (i.e. n-1).

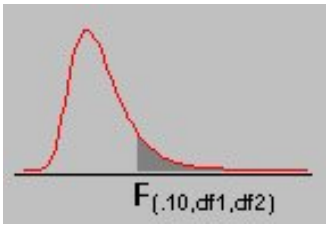


Degree of Freedom(v)	Upper-tail area α										
	0.25	**0.2**	**0.15**	**0.1**	**0.05**	**0.025**	**0.01**	**0.005**	**0.0025**	**0.001**	**0.0005**
1	1.000	1.376	1.963	3.078	6.314	12.710	31.820	63.660	127.300	318.300	636.600
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	14.090	22.330	31.600
3	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	7.453	10.210	12.920
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
120	0.677	0.845	1.041	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291



Upper-tail area α

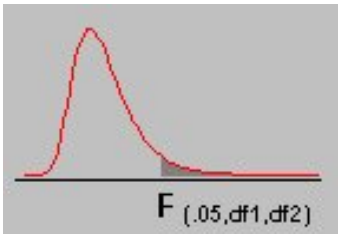
df	0.995	0.99	0.975	0.95	0.9	0.75	0.5	0.25	0.1	0.05	0.025	0.02	0.01	0.005
1	0.000	0.000	0.001	0.004	0.016	0.102	0.455	1.323	2.706	3.841	5.024	5.412	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	0.575	1.386	2.773	4.605	5.991	7.378	7.824	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	1.213	2.366	4.108	6.251	7.815	9.348	9.837	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	1.923	3.357	5.385	7.779	9.488	11.143	11.668	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	2.675	4.351	6.626	9.236	11.071	12.833	13.388	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	3.455	5.348	7.841	10.645	12.592	14.449	15.033	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	4.255	6.346	9.037	12.017	14.067	16.013	16.622	18.475	20.278
8	1.344	1.647	2.180	2.733	3.490	5.071	7.344	10.219	13.362	15.507	17.535	18.168	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	5.899	8.343	11.389	14.684	16.919	19.023	19.679	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	6.737	9.342	12.549	15.987	18.307	20.483	21.161	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	7.584	10.341	13.701	17.275	19.675	21.920	22.618	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	8.438	11.340	14.845	18.549	21.026	23.337	24.054	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	9.299	12.340	15.984	19.812	22.362	24.736	25.472	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	10.165	13.339	17.117	21.064	23.685	26.119	26.873	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	11.037	14.339	18.245	22.307	24.996	27.488	28.259	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	11.912	15.339	19.369	23.542	26.296	28.845	29.633	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	12.792	16.338	20.489	24.769	27.587	30.191	30.995	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	13.675	17.338	21.605	25.989	28.869	31.526	32.346	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	14.562	18.338	22.718	27.204	30.144	32.852	33.687	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	15.452	19.337	23.828	28.412	31.410	34.170	35.020	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	16.344	20.337	24.935	29.615	32.671	35.479	36.343	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	17.240	21.337	26.039	30.813	33.924	36.781	37.659	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	18.137	22.337	27.141	32.007	35.172	38.076	38.968	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	19.037	23.337	28.241	33.196	36.415	39.364	40.270	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	19.939	24.337	29.339	34.382	37.652	40.646	41.566	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	20.843	25.336	30.435	35.563	38.885	41.923	42.856	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	21.749	26.336	31.528	36.741	40.113	43.195	44.140	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	22.657	27.336	32.620	37.916	41.337	44.461	45.419	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	23.567	28.336	33.711	39.087	42.557	45.722	46.693	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	24.478	29.336	34.800	40.256	43.773	46.979	47.962	50.892	53.672



F Table for alpha=.10 .

df for numerator(v_1)

df for denominator(v_2)	1	2	3	4	5	6	7	8	9	10	12
1	39.863	49.500	53.593	55.833	57.240	58.204	58.906	59.439	59.858	60.195	60.705
2	8.526	9.000	9.162	9.243	9.293	9.326	9.349	9.367	9.381	9.392	9.408
3	5.538	5.462	5.391	5.343	5.309	5.285	5.266	5.252	5.240	5.230	5.216
4	4.545	4.325	4.191	4.107	4.051	4.010	3.979	3.955	3.936	3.920	3.896
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297	3.268
6	3.776	3.463	3.289	3.181	3.108	3.055	3.014	2.983	2.958	2.937	2.905
7	3.589	3.257	3.074	2.961	2.883	2.827	2.785	2.752	2.725	2.703	2.668
8	3.458	3.113	2.924	2.806	2.726	2.668	2.624	2.589	2.561	2.538	2.502
9	3.360	3.006	2.813	2.693	2.611	2.551	2.505	2.469	2.440	2.416	2.379
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323	2.284
11	3.225	2.860	2.660	2.536	2.451	2.389	2.342	2.304	2.274	2.248	2.209
12	3.177	2.807	2.606	2.480	2.394	2.331	2.283	2.245	2.214	2.188	2.147
13	3.136	2.763	2.560	2.434	2.347	2.283	2.234	2.195	2.164	2.138	2.097
14	3.102	2.726	2.522	2.395	2.307	2.243	2.193	2.154	2.122	2.095	2.054
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059	2.017
16	3.048	2.668	2.462	2.333	2.244	2.178	2.128	2.088	2.055	2.028	1.985
17	3.026	2.645	2.437	2.308	2.218	2.152	2.102	2.061	2.028	2.001	1.958
18	3.007	2.624	2.416	2.286	2.196	2.130	2.079	2.038	2.005	1.977	1.933
19	2.990	2.606	2.397	2.266	2.176	2.109	2.058	2.017	1.984	1.956	1.912
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937	1.892
21	2.961	2.575	2.365	2.233	2.142	2.075	2.023	1.982	1.948	1.920	1.875
22	2.949	2.561	2.351	2.219	2.128	2.061	2.008	1.967	1.933	1.904	1.859
23	2.937	2.549	2.339	2.207	2.115	2.047	1.995	1.953	1.919	1.890	1.845
24	2.927	2.538	2.327	2.195	2.103	2.035	1.983	1.941	1.906	1.877	1.832
25	2.918	2.528	2.317	2.184	2.092	2.024	1.971	1.929	1.895	1.866	1.820
26	2.909	2.519	2.307	2.174	2.082	2.014	1.961	1.919	1.884	1.855	1.809
27	2.901	2.511	2.299	2.165	2.073	2.005	1.952	1.909	1.874	1.845	1.799
28	2.894	2.503	2.291	2.157	2.064	1.996	1.943	1.900	1.865	1.836	1.790
29	2.887	2.495	2.283	2.149	2.057	1.988	1.935	1.892	1.857	1.827	1.781
30	2.881	2.489	2.276	2.142	2.049	1.980	1.927	1.884	1.849	1.819	1.773
40	2.835	2.440	2.226	2.091	1.997	1.927	1.873	1.829	1.793	1.763	1.715
60	2.791	2.393	2.177	2.041	1.946	1.875	1.819	1.775	1.738	1.707	1.657
120	2.748	2.347	2.130	1.992	1.896	1.824	1.767	1.722	1.684	1.652	1.601



F Table for alpha=.05 .

df for denominator(v_2)	df for numerator(v_1) 1	2	3	4	5	6	7	8	9	10	12
1	161.448	199.500	215.707	224.583	230.162	233.986	236.768	238.883	240.543	241.882	243.906
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396	19.413
3	10.128	9.552	9.277	9.117	9.014	8.941	8.887	8.845	8.812	8.786	8.745
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.912
5	6.608	5.786	5.410	5.192	5.050	4.950	4.876	4.818	4.773	4.735	4.678
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	4.000
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.575
8	5.318	4.459	4.066	3.838	3.688	3.581	3.501	3.438	3.388	3.347	3.284
9	5.117	4.257	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.073
10	4.965	4.103	3.708	3.478	3.326	3.217	3.136	3.072	3.020	2.978	2.913
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854	2.788
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.687
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671	2.604
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602	2.534
15	4.543	3.682	3.287	3.056	2.901	2.791	2.707	2.641	2.588	2.544	2.475
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494	2.425
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450	2.381
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.342
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378	2.308
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.278
21	4.325	3.467	3.073	2.840	2.685	2.573	2.488	2.421	2.366	2.321	2.250
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297	2.226
23	4.279	3.422	3.028	2.796	2.640	2.528	2.442	2.375	2.320	2.275	2.204
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255	2.183
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405	2.337	2.282	2.237	2.165
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388	2.321	2.266	2.220	2.148
27	4.210	3.354	2.960	2.728	2.572	2.459	2.373	2.305	2.250	2.204	2.132
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190	2.118
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346	2.278	2.223	2.177	2.105
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.092
40	4.085	3.232	2.839	2.606	2.450	2.336	2.249	2.180	2.124	2.077	2.004
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.917
120	3.920	3.072	2.680	2.447	2.290	2.175	2.087	2.016	1.959	1.911	1.834

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5808	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6514	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	3	4	5	6	6	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7083	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7318	7226	7235	1	2	3	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7350	7355	73	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7619	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	1	1	2	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9634	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9982	9917	9921	9928	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1018	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1333	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1364	1368	1371	1374	1377	0	1	1	1	2	2	2	2	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	2	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	2	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1492	1493	1496	1500	1503	1507	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	2	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	2	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2245	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2376	2381	2386	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2465	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	8
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	8
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	8

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	8
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	8
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	8
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4256	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5376	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	11
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	11
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8895	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20